9. INTEGRALE MULTIPLE

9.2. Exerciţii rezolvate

Exerciţiul 9.2.1. Să se calculeze integralele:

a) $\iint_D \frac{x^2}{1+y^2}dydx$, unde D = {(x, y) $\in \mathbb{R}^2$| 0 ≤ x ≤ 1, 0 ≤ y ≤ 1}

b) $\iint_D \frac{x^2}{y^2}dydx$, unde D = {(x, y) $\in \mathbb{R}^2$| 1 ≤ x ≤ 2, $\frac{1}{x}$ ≤ y ≤ x}

Soluţie. a) $\iint_D \frac{x^2}{1+y^2}dydx = \int_0^1\left[\int_0^1 \frac{x^2}{1+y^2}dy\right]dx = = \int_0^1\left[x^2 \cdot arctg\ y \Big|_{y=0}^{y=1}\right]dx =$

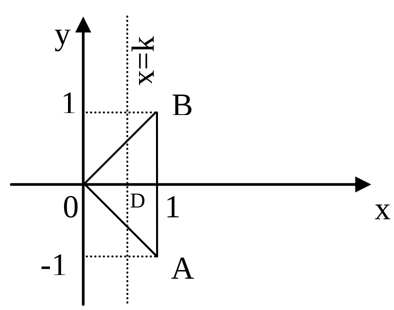
$$\int_0^1 x^2\cdot\frac{\pi}{4}dx = \frac{\pi}{4}\cdot\frac{x^3}{3}\Bigg|_{x=0}^{x=1} = \frac{\pi}{12}$$

b) $\iint_D \frac{x^2}{y^2}dydx = \int_1^2\left[\int_{1/x}^{x}\frac{x^2}{y^2}dy\right]dx = \int_1^2\left[-\frac{x^2}{y}\Bigg|_{y=\frac{1}{x}}^{y=x}\right]dx =$

$$= \int_1^2(-x+x^3)dx = \left(-\frac{x^2}{2}+\frac{x^4}{4}\right)\Bigg|_{x=1}^{x=2} = \frac{9}{4}$$

Exerciţiul 9.2.2. Să se calculeze $\iint_D \sqrt{x^2-y^2}\,dxdy$, unde D este triunghiul cu vârfurile O(0, 0), A(1, -1) şi B(1, 1).

Soluţie. Domeniul D este simplu în raport cu axa Oy (vezi figura) deoarece o dreaptă x = k, k ∈ (0, 1) intersectează pe D după un interval.



Dreptele OA şi OB au ecuaţiile:

OA: $\frac{y-0}{-1-0} = \frac{x-0}{1-0}$, adică OA: - y = x

OB: $\frac{y-0}{1-0} = \frac{x-0}{1-0}$, adică OB: y = x.

Deci: OA: y = - x

OB: y = x

Atunci domeniul D pe care se calculează integrala dublă este

D = {(x, y) $\in \mathbb{R}^2$| 0 ≤ x ≤ 1, -x ≤ y ≤ x}

Putem aplica deci formula din exercițiul 11.2.2. pentru a = 0, b = 1,
$\varphi_1(x) = -x$, $\varphi_2(x) = x$.

Avem $$\iint_D \sqrt{x^2 - y^2}\,dxdy = \int_0^1 \left[\int_{-x}^{x} \sqrt{x^2 - y^2}\,dy\right] dx.$$

Calculăm întâi $F(x) = \int_{-x}^{x} \sqrt{x^2 - y^2}\,dy$. Observăm că funcția

$g(y) = \sqrt{x^2 - y^2}$ este pară, adică $g(-y) = g(y)$. Atunci rezultă:

$$F(x) = 2\int_0^x \sqrt{x^2 - y^2}\,dy = 2\int_0^x \frac{x^2 - y^2}{\sqrt{x^2 - y^2}}\,dy =$$

$$= 2x^2 \arcsin \frac{y}{x}\Big|_{y=0}^{y=x} + 2\int_0^x y \cdot \left(\sqrt{x^2 - y^2}\right)'\,dy =$$

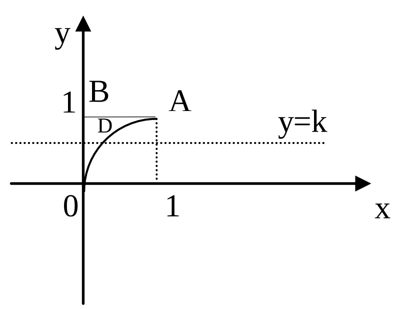
$$= 2x^2 \cdot \frac{\pi}{2} + 2y\sqrt{x^2 - y^2}\Big|_{y=0}^{y=x} - 2\int_0^x \sqrt{x^2 - y^2}\,dy = \pi x^2 - F(x)$$

Deci $F(x) = \pi x^2 - F(x)$, de unde $F(x) = \dfrac{\pi x^2}{2}$.

Așadar $$\iint_D \sqrt{x^2 - y^2}\,dxdy = \int_0^1 F(x)\,dx = \int_0^1 \frac{\pi x^2}{2}\,dx = \frac{\pi}{2} \cdot \frac{x^3}{3}\Big|_0^1 = \frac{\pi}{6}.$$

Exercițiul 9.2.3. Să se calculeze $\iint_D e^{\frac{x}{y}}\,dxdy$, unde D este triunghiul OAB, limitat de parabola $y^2 = x$ și dreptele $x = 0$, $y = 1$.

Soluție. Domeniul D este simplu în raport cu axa Ox (vezi figura) deoarece o dreaptă $y = k$, $k \in (0, 1)$, intersectează pe D după un interval.



Domeniul D este caracterizat de :

$$D = \{(x, y) \in \mathbb{R}^2 \mid 0 \le y \le 1,\ 0 \le x \le y^2\}$$

Aplicăm formula din exercițiul 11.2.3. pentru c = 0, d = 1, $\psi_1(y) = 0$,
$\psi_2(y) = y^2$.

Avem deci $$\iint_D e^{\frac{x}{y}}\,dxdy = \int_0^1 \left[\int_0^{y^2} e^{\frac{x}{y}}\,dx\right] dy.$$

Calculăm F(y) = $\int_0^{y^2} e^{\frac{x}{y}}dx = e^{\frac{x}{y}} \cdot y \Big|_{x=0}^{x=y^2} = e^y y - y$ şi deci:

$$\iint_D e^{\frac{x}{y}}dxdy = \int_0^1 F(y)dy = \int_0^1 (ye^y - y)dy = \int_0^1 y(e^y)'dy - \frac{y^2}{2}\Big|_0^1 =$$

$$= ye^y\Big|_0^1 - \int_0^1 e^y dy - \frac{1}{2} = e - e^y\Big|_0^1 - \frac{1}{2} = \frac{1}{2}$$

Exerciţiul 9.2.4. Să se calculeze următoarele integrale duble, pe domeniile indicate:

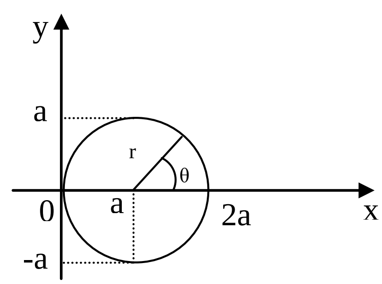
a) $\iint_D (x^2 + y^2)dxdy$, D fiind domeniul limitat de cercul de ecuaţie $x^2+y^2= 2ax$;

b) $\iint_D \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}dxdy$, D fiind domeniul limitat de elipsa de ecuaţie $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$;

c) $\iint_D (x^2 + y^2) \cdot ydxdy$, D fiind domeniul limitat de axa Ox şi de porţiunea din cardioida r = a(a + cosθ), situată deasupra axei Ox.

Soluţii. a) Ecuaţia cercului ce limitează domeniul D se mai poate scrie:
$(x - a)^2 + y^2 = a^2$, deci ea defineşte cercul cu centrul în punctul de coordonate (a, 0) şi de rază a. Este convenabil să folosim coordonatele polare pentru calculul integralei duble date.

Facem aşadar schimbarea de variabile (x, y) → (r, θ), dată prin transformarea $\begin{cases} x - a = r\cos\theta \\ y = r\sin\theta \end{cases}$



Noul domeniu de integrare (domeniul transformat) este:

$$D^* = \{(x, y) \in \mathbb{R}^2 | 0 \le r \le a, 0 \le \theta \le 2\pi\}.$$

Jacobianul acestei transformări este:

$$J = \frac{D(x,y)}{D(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r \cdot \sin\theta \\ \sin\theta & r \cdot \cos\theta \end{vmatrix} = r\cdot\cos^2\theta + r\cdot\sin^2\theta = r,$$

iar $x^2 + y^2 = a^2 + 2ar\cos\theta + r^2$.

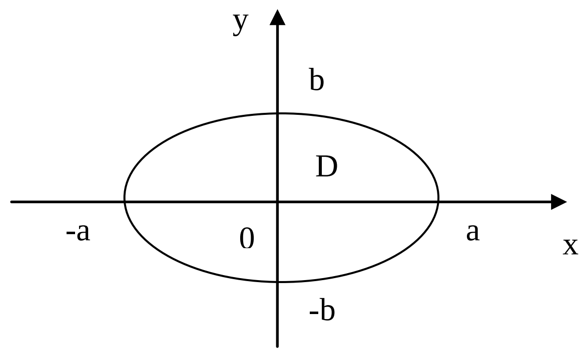
Deci integrala de calculat devine:

$$\iint_{D^*} (a^2 + 2ar\cos\theta + r^2) r dr d\theta = \int_0^a \left[\int_0^{2\pi} (a^2 r + 2ar^2 \cos\theta + r^3) d\theta \right] dr =$$

$$\int_0^a \left[(a^2 r + r^3) \cdot \theta \Big|_0^{2\pi} \right] dr + \int_0^a \left[2ar^2 \cdot \sin\theta \Big|_{\theta=0}^{\theta=2\pi} \right] dr =$$

$$= 2\pi \int_0^a (a^2 r + r^3) dr = 2\pi \left(a^2 \cdot \frac{r^2}{2} + \frac{r^4}{4} \right) \Bigg|_{r=0}^{r=a} = \frac{3\pi a^4}{2} .$$

b)Trecem la coordonate polare generalizate: $\begin{cases} x = a\, r \cos\theta \\ y = b\, r \sin\theta \end{cases}$



Domeniul transformat este: $D^* = \{(r, \theta) \in \mathbb{R}^2 \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$.

Jacobianul transformării este:

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} a \cdot \cos\theta & -ar \cdot \sin\theta \\ b \cdot \sin\theta & br \cdot \cos\theta \end{vmatrix} = abr ,$$

iar $\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} = \sqrt{1 - r^2}$.

Aşadar, integrala devine:

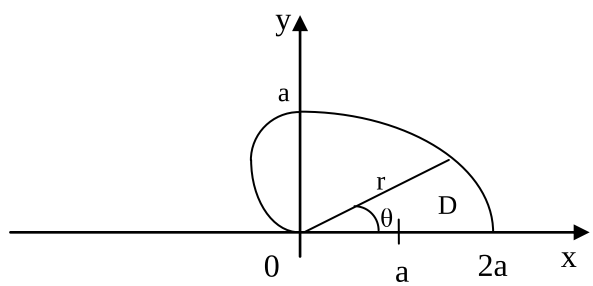
$$\iint_{D^*} abr\sqrt{1 - r^2}\, dr d\theta = \int_0^1 \left[\int_0^{2\pi} abr\sqrt{1 - r^2}\, d\theta \right] dr =$$

$$ab \int_0^1 \left[r\sqrt{1 - r^2} \cdot \theta \Big|_{\theta=0}^{\theta=2\pi} \right] dr = ab \int_0^1 2\pi r\sqrt{1 - r^2}\, dr = 2\pi ab \int_0^1 r\sqrt{1 - r^2}\, dr$$

$$= -\pi ab \int_0^1 (1 - r^2)' \sqrt{1 - r^2}\, dr = -\pi ab (1 - r^2)^{3/2} \cdot \frac{2}{3} \Bigg|_0^1 = \frac{2\pi ab}{3} .$$

c) Trecem la coordonate polare: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$. Domeniul pe care se face integrarea este D (vezi figura), iar D^* este :

$$D^* = \{(r, \theta) \mid 0 \le \theta \le \pi,\ 0 \le r \le a(1 + \cos\theta)\}.$$



Avem $(x^2 + y^2)y = r^3\sin\theta$ şi J = r. Deci:

$$\iint_D (x^2+y^2)\cdot y dxdy = = \int_0^{\pi}\left[\int_0^{a(1+\cos\theta)} r^4 \sin\theta dr\right]d\theta = \int_0^{\pi}\left[\frac{r^5}{5}\sin\theta\Big|_{r=0}^{r=a(1+\cos\theta)}\right]d\theta =$$

$$= \frac{a^5}{5}\int_0^{\pi}\left[(1+\cos\theta)^5\sin\theta\right]d\theta = \frac{a^5}{5}\left[\frac{-(1+\cos\theta)^6}{6}\right]\Bigg|_0^{\pi} = \frac{32a^5}{15}$$

Exerciţiul 9.2.5. Să se calculeze aria interiorului elipsei de ecuaţie:

$$(x - 2y +3)^2 + (3x + 4y -1)^2 = 100$$

Soluţie. Folosim formula: Aria (D) = $\iint_D dxdy$, unde D este interiorul elipsei.

Efectuăm schimbarea de variabilă (x, y) →(u, v) dată prin:

$\begin{cases} x-2y=u \\ 3x+4y=v \end{cases}$, (u, v) $\in D^*$

$$D^*=\{(u,v) \in \mathbb{R}^2 | (u+3)^2 + (v-1)^2 \le 100\}$$

Jacobianul acestei transformări este:

$$J^* = \frac{D(x,y)}{D(u,v)} = \left[\frac{D(u,v)}{D(x,y)}\right]^{-1} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & -2 \\ 3 & 4 \end{vmatrix}^{-1} = \frac{1}{10}$$

Deci: Aria(D) = $\iint_{D^*} J^* dudv = \iint_{D^*} \frac{1}{10} dudv = \frac{1}{10}\iint_{D^*} dudv$.

Pentru calculul acestei din urmă integrale trecem la coordonate polare: $\begin{cases} u+3=r\cos\theta \\ v-1=r\sin\theta \end{cases}$, unde (r, θ) $\in D^{**}$,

$$D^{**} = \{(r, \theta) | 0 \le r \le 10, 0 \le \theta \le 2\pi\}.$$

Jacobianul transformării este în acest caz J = r, iar

$$\iint_{D^*} dudv = \iint_{D^*} Jdrd\theta = \int_0^{10}\left[\int_0^{2\pi} rd\theta\right]dr = \int_0^{10}\left(r\cdot d\theta\Big|_{\theta=0}^{\theta=2\pi}\right)dr =$$

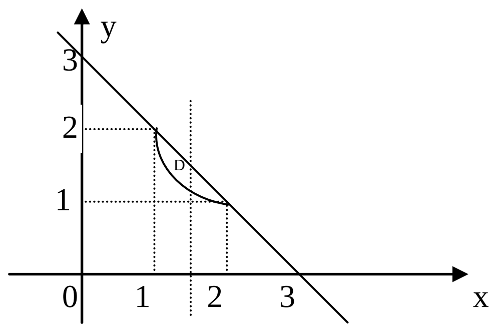
$$= \int_0^{10} r\cdot 2\pi dr = 2\pi\cdot\frac{r^2}{2}\Bigg|_{r=0}^{r=10} = 100\pi$$

Deci: Aria(D) = $\frac{1}{10}\iint_{D^*} dudv = 10\pi$.

Exerciţiul 9.2.6. Să se calculeze masa unei plăci plane D, limitate de
$x + y = 3$, $xy = 2$ şi a cărei densitate este $\rho(x, y) = xy$.

Soluţie. M = $\iint_D \rho(x,y)dxdy = \iint_D xydxdy$. Domeniul D poate fi caracterizat astfel (aşa cum se vede din figură):

$$D = \{(x, y) \in \mathbb{R}^2 | \ 1 \le x \le 2,\ \frac{2}{x} \le y \le 3 - x\}$$

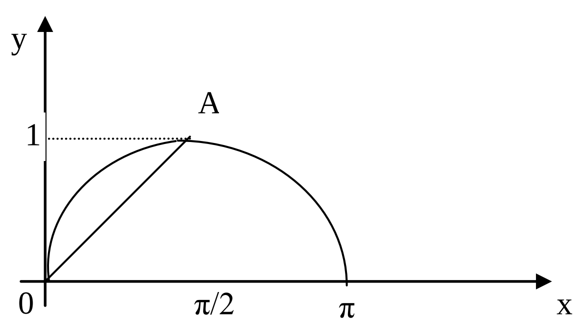


Atunci:

$$M = \int_1^2 \int_{2/x}^{3-x} xydxdy = \int_1^2 \left[x \cdot \frac{y^2}{2} \Bigg|_{y=\frac{2}{x}}^{y=3-x} \right] dx = \int_1^2 \left[\frac{9x - 6x^2 + x^3}{2} - \frac{2}{x} \right] dx$$

$$= \left(\frac{9}{2} \cdot \frac{x^2}{2} - 3 \cdot \frac{x^3}{3} + \frac{1}{2} \cdot \frac{x^4}{4} - 2\ln x \right) \Bigg|_1^2 = \frac{18}{3} - 2\ln 2 .$$

Exerciţiul 9.2.7. Să se calculeze coordonatele centrului de greutate al plăcii plane omogene din figura de mai jos, limitat de curba $y = \sin x$ şi dreapta OA care trece prin origine şi punctul $A\left(\frac{\pi}{2}, 1\right)$.



Soluţie. Dreapta OA are ecuaţia OA: $y = \frac{2x}{\pi}$. Deci,

$$D = \{(x, y) \in \mathbb{R}^2 | 0 \le x \le \frac{\pi}{2},\ \frac{2x}{\pi} \le y \le \sin x\}$$

Se calculează M = $\iint_D \rho(x,y)dxdy = k\iint_D dxdy$, unde $\rho(x, y) = k = \text{const}$ fiind vorba de o placă omogenă.

Avem:

$$\iint_D dxdy = \int_0^{\pi/2}\left[\int_{2x/\pi}^{\sin x} dy\right]dx = \int_0^{\pi/2}\left(\sin x - \frac{2x}{\pi}\right)dx = 1-\frac{\pi}{4},$$

şi deci M = k $\left(1-\frac{\pi}{4}\right)$.

Pe de altă parte,

$$\iint_D x\rho(x,y)dxdy = \text{k} \iint_D xdxdy$$

$$= \text{k} \int_0^{\pi/2}\left[\int_{2x/\pi}^{\sin x} xdy\right]dx = k\int_0^{\pi/2} x\left(\sin x - \frac{2x}{\pi}\right)dx$$

$$= \text{k} \int_0^{\pi/2} x\sin xdx - \frac{2k}{\pi}\cdot\frac{x^3}{3}\Big|_0^{\pi/2}$$

$$= \text{- kx} \cdot \cos x\Big|_0^{\pi/2} + k\int_0^{\pi/2}\cos xdx - \frac{2k}{\pi}\cdot\frac{\pi^3}{24} = \text{k}\cdot \sin x\Big|_0^{\pi/2} - \frac{k\pi^2}{12}$$

$$= \text{k} - \frac{k\pi^2}{12} = k\left(1-\frac{\pi^2}{12}\right).$$

Deci $x_G = \frac{1}{M}\iint_D x\rho(x,y)dxdy = \frac{k\left(1-\frac{\pi^2}{12}\right)}{k\left(1-\frac{\pi}{4}\right)} = \frac{12-\pi^2}{3(4-\pi)}$.

$$\iint_D y\rho(x,y)dxdy = \text{k} \iint_D ydxdy = \int_0^{\pi/2}\left[\int_{2x/\pi}^{\sin x} ydy\right]dx = k\int_0^{\pi/2}\left[\frac{y^2}{2}\Big|_{y=\frac{2x}{\pi}}^{y=\sin x}\right]dx =$$

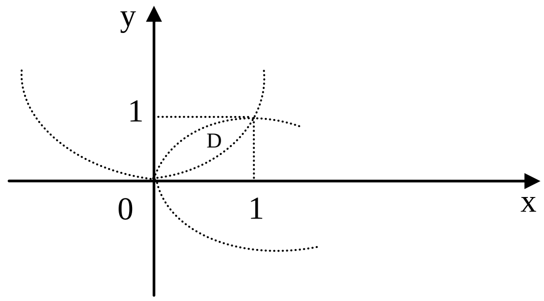
$$= \frac{k}{2}\int_0^{\pi/2}\left(\sin^2 x - \frac{4x^2}{\pi^2}\right)dx = \frac{k}{2}\left[\int_0^{\pi/2}\frac{1-\cos 2x}{2}dx - \frac{4}{\pi^2}\cdot\frac{x^3}{3}\Big|_0^{\pi/2}\right]$$

$$= \frac{k}{2}\left[\left(\frac{x}{2}-\frac{\sin 2x}{4}\right)\Big|_0^{\pi/2} - \frac{\pi}{6}\right] = \frac{k}{2}\cdot\frac{\pi}{12} = \frac{k\pi}{24}$$

Deci $y_G = \frac{1}{M}\iint_D y\rho(x,y)dxdy = \frac{\frac{k\pi}{24}}{k\left(1-\frac{\pi}{4}\right)} = \frac{\pi}{6(4-\pi)}$.

Exerciţiul 9.2.8. Să se calculeze momentele de inerţie în raport cu axele de coordonate pentru placa omogenă mărginită de curbele $y = x^2$, $x = y^2$.

Soluţie.



Domeniul D este caracterizat de:

$$D = \{(x, y) \in \mathbb{R}^2 | 0 \le x \le 1,\ x^2 \le y \le \sqrt{x}\ \}$$

Deci $I_x = \iint_D y^2 \rho(x,y)dxdy = k\iint_D y^2 dxdy = k\int_0^1 \left[\int_{x^2}^{\sqrt{x}} y^2 dy\right]dx =$

$= k\int_0^1 \left(\left.\frac{y^3}{3}\right|_{y=x^2}^{y=\sqrt{x}}\right)dx = k\int_0^1 \left(\frac{x^{3/2}}{3} - \frac{x^6}{3}\right)dx = k\left.\left(\frac{x^{5/2}}{3}\cdot\frac{2}{5} - \frac{x^7}{21}\right)\right|_0^1 dx =$

$= \frac{3k}{35}$.

Analog $I_y = \iint_D x^2 \rho(x,y)dxdy = k\iint_D x^2 dxdy = k\int_0^1 \left[\int_{x^2}^{\sqrt{x}} x^2 dy\right]dx =$

$= k\int_0^1 x^2(\sqrt{x} - x^2)dx = k\int_0^1 (x^{5/2} - x^4)dx = \frac{3k}{35}$.

Exerciţiul 9.2.9. Să se calculeze următoarele integrale:

a) $\iiint_\Omega \frac{1}{\sqrt{x+y+z+1}}dxdydz$, unde

$$\Omega = \{(x, y, z) \in \mathbb{R}^3 | 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$$

b) $\iiint_\Omega xyz dxdydz$, unde

$$\Omega = \{(x, y, z) \in \mathbb{R}^3 | 0 \le z \le 1 - x - y,\ 0 \le y \le 1 - x,\ 0 \le x \le 1\}$$

Soluţii. a) Avem:

$$\iiint_\Omega \frac{1}{\sqrt{x+y+z+1}}\,dxdydz = \int_0^1\left[\int_0^1\left[\int_0^1 \frac{1}{\sqrt{x+y+z+1}}\,dx\right]dy\right]dz =$$

$$\int_0^1\left[\int_0^1\left[(x+y+z+1)^{1/2}\cdot 2\Big|_{x=0}^{x=1}\right]dy\right]dz =$$

$$= 2\int_0^1\left[\int_0^1\left[(y+z+2)^{1/2}-(y-z+1)^{1/2}\right]dy\right]dz =$$

$$= 2\int_0^1\left[(y+z+2)^{3/2}\cdot\frac{2}{3}-(y+z+1)^{3/2}\cdot\frac{2}{3}\Big|_{y=0}^{y=1}\right]dz =$$

$$= \frac{4}{3}\int_0^1\left[(z+3)^{3/2}-(z+2)^{3/2}-(z+2)^{3/2}+(z+1)^{3/2}\right]dz =$$

$$= \frac{4}{3}\left[(z+3)^{5/2}\,\frac{2}{5}-(z+2)^{5/2}\,\frac{2}{5}-(z+2)^{5/2}\,\frac{2}{5}+(z+1)^{5/2}\,\frac{2}{5}\right]\Bigg|_{z=0}^{z=1}$$

$$= \frac{4}{3}\cdot\frac{2}{5}\left[4^{5/2}-3^{5/2}-3^{5/2}+2^{5/2}-3^{5/2}+2^{5/2}+2^{5/2}-1\right] =$$

$$= \frac{8}{15}\left[4^{5/2}-3^{7/2}+3\cdot 2^{5/2}-1\right] = \frac{8}{15}\left(31+12\sqrt{2}-27\sqrt{3}\right).$$

b) $\displaystyle\iiint_\Omega xyzdxdydz = \iint_D\left[\int_0^{1-x-y} xyzdz\right]dxdy = \iint_D xy\frac{z^2}{2}\Bigg|_{z=0}^{z=1-x-y} dxdy =$

$$\frac{1}{2}\iint_D xy(1-x-y)^2\,dxdy = \frac{1}{2}\iint_D xy(1+x^2+y^2-2x-2y+2xy)dxdy$$

$$= \frac{1}{2}\iint_D (xy+x^3y+xy^3-2x^2y-2xy^2+2x^2y^2)dxdy,$$

unde D = $\{(x, y) \in \mathbb{R}^2 : 0 \le y \le 1 - x,\ 0 \le x \le 1\}$

Prin urmare,

$$\iiint_\Omega xyzdxdydz = \frac{1}{2}\int_0^1\left[\int_0^{1-x}(xy+x^3y+xy^3-2x^2y-2xy^2+2x^2y^2)dy\right]dx$$

$$= \frac{1}{2}\int_0^1\left[\left(x\frac{y^2}{2}+x^3\,\frac{y^2}{2}+x\frac{y^4}{4}-x^2y^2-2x\frac{y^3}{3}+2x^2\,\frac{y^3}{3}\right)\Bigg|_{y=0}^{y=1-x}\right]dx$$

$$\frac{1}{2}\int_0^1\left[\frac{x}{2}(1-x)^2+\frac{x^3}{2}(1-x)^2+\frac{x}{4}(1-x)^4-x^2(1-x)^2-\frac{2x}{3}(1-x)^3\right]dx =$$

$$\frac{1}{24}\int_0^1(x-4x^2+6x^3-4x^4+x^5)dx =$$

$$= \frac{1}{24}\left(\frac{x^2}{2} - 4\frac{x^3}{3} + 6\frac{x^4}{4} - 4\frac{x^5}{5} + \frac{x^6}{6}\right)\Bigg|_0^1 = \frac{1}{720}$$

Exercițiul 9.2.10. Să se calculeze următoarele integrale:

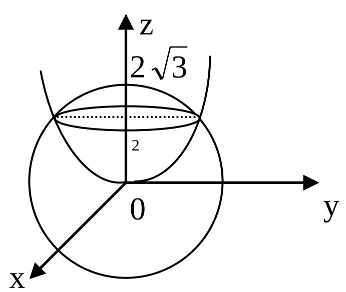
a) $\iiint\limits_{\Omega} (x^2 + y^2 + z^2)dxdydz$, unde Ω este domeniul mărginit de sfera $x^2 + y^2 + z^2 = 12$ şi paraboloidul $x^2 + y^2 = 4z$;

b) $\iiint\limits_{\Omega} \sqrt{x^2 + y^2}\,dxdydz$, unde

$$\Omega = \{(x, y, z) \in \mathbb{R}^3:\ x^2 + y^2 \le 9,\ z \ge 0,\ x + y + z \le 6\}$$

Soluţii. a) Cele două suprafeţe se intersectează după cercul:

(C): $\begin{cases} x^2 + y^2 = 8 \\ z = 2 \end{cases}$. Evident $0 \le z \le 2\sqrt{3}$



Aplicăm deci formula:

$$\iiint\limits_{\Omega} (x^2 + y^2 + z^2)dxdydz = \int_0^{2\sqrt{3}} \left[\iint\limits_{D_z} (x^2 + y^2 + z^2)dxdy \right] dz,$$

unde D_z este proiecţia pe planul xOy a unei secţiuni făcute în Ω cu un plan $z = z_0$, $z_0 \in [0, 2\sqrt{3}]$. D_z, este caracterizat de:

(D_z'): $x^2 + y^2 \le 4z$, dacă $z \in [0, 2]$ şi

(D_z''): $x^2 + y^2 \le 12 - z^2$, dacă $z \in [2, 2\sqrt{3}]$

Deci $\iiint\limits_{\Omega} (x^2 + y^2 + z^2)dxdydz = \int_0^2 \left[\iint\limits_{D_z'} (x^2 + y^2 + z^2)dxdy \right] dz +$

$$+ \int_2^{2\sqrt{3}} \left[\iint\limits_{D_z''} (x^2 + y^2 + z^2)dxdy \right] dz$$

Pentru calculul integralelor duble folosim coordonatele polare, întrucât (D_z') şi (D_z'') sunt discuri.

Pentru prima integrală dublă avem:

$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$, $r \in [0, 2\sqrt{z}]$, $\theta \in [0, 2\pi]$, iar jacobianul este J = r.

Deci,

$$\iint\limits_{D_z'} (x^2+y^2+z^2)dxdy = \int_0^{2\sqrt{z}}\left[\int_0^{2\pi}(r^2+z^2)\cdot r d\theta\right]dr = 2\pi\int_0^{2\sqrt{z}} r(r^2+z^2)dr =$$

$$2\pi\left[\frac{r^4}{4}+z^2\cdot\frac{r^2}{2}\right]\Bigg|_{r=0}^{r=2\sqrt{z}} = 4\pi z^2(2+z).$$

Pentru cea de a doua integrală dublă, avem:

$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$, $r\in[0, \sqrt{12-z^2}]$, $\theta\in[0, 2\pi]$.

Deci,

$$\iint\limits_{D_z''} (x^2+y^2+z^2)dxdy = \int_0^{\sqrt{12-z^2}}\left[\int_0^{2\pi}(r^2+z^2)\cdot r d\theta\right]dr =$$

$$= 2\pi\int_0^{\sqrt{12-z^2}} r(r^2+z^2)dr = \frac{\pi}{2}(144-z^4)$$

Aşadar,

$$\iiint\limits_{\Omega}(x^2+y^2+z^2)dxdydz = \int_0^2 4\pi z^2(2+z)dz + \int_2^{2\sqrt{3}}\frac{\pi}{2}(144-z^4)dz = \frac{32\pi}{5}\left(18\sqrt{3}-\frac{97}{6}\right).$$

b) Suntem îm situaţia a doua de la exerciţiul 9.3.2. pentru $D = \{(x, y) \in \mathbb{R}^2\colon\ x^2+y^2 \le 9\}$, $\varphi_1(x, y) = 0$ şi $\varphi_2(x, y) = 6 - x - y$

Aplicăm deci formula adecvată, adică:

$$\iiint\limits_{\Omega}\sqrt{x^2+y^2}\,dxdydz = \iint\limits_{D}\left[\int_{\varphi_1(x,y)}^{\varphi_2(x,y)}\sqrt{x^2+y^2}\,dz\right]dxdy.$$

Deci

$$\iiint\limits_{\Omega}\sqrt{x^2+y^2}\,dxdydz = \iint\limits_{D}\left[\int_0^{6-x-y}\sqrt{x^2+y^2}\,dz\right]dxdy$$

$$= \iint\limits_{D}\sqrt{x^2+y^2}\cdot(6-x-y)dxdy$$

Calculăm această integrală prin trecere la coordonate polare. Avem:

$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$, $r \in [0, 3]$; $\theta\in[0, 2\pi]$.

$$\iint\limits_{D}\sqrt{x^2+y^2}\cdot(6-x-y)dxdy = \int_0^{2\pi}\left[\int_0^3 r^2\cdot(6-r\cos\theta-r\sin\theta)dr\right]d\theta =$$

$$\int_0^{2\pi}\left[\left(6\cdot\frac{r^3}{3}-\frac{r^4}{4}\cos\theta-\frac{r^4}{4}\sin\theta\right)\Bigg|_{r=0}^{r=3}\right]d\theta =$$

$$\int_0^{2\pi}\left[54-\frac{3^4}{4}\cos\theta-\frac{3^4}{4}\sin\theta\right]d\theta = 108\pi$$

Exerciţiul 9.2.11. Să se calculeze următoarele integrale triple:

a) $\iiint\limits_{\Omega} (x^2 + y^2 + z^2)dxdydz$, unde

$\Omega = \{(x, y, z) \in \mathbb{R}^3: y^2 + z^2 \le x^2, x^2 + y^2 + z^2 \le 4, x \ge 0\}$;

b) $\iiint\limits_{\Omega} zdxdydz$, unde Ω este domeniul limitat de conul

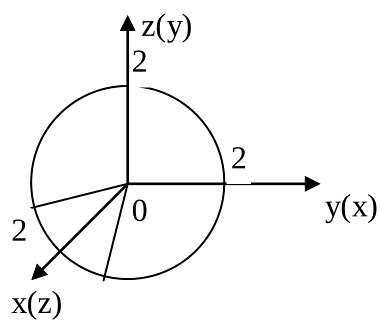
$$z^2 = \frac{h^2}{R^2}(x^2 + y^2) \text{ şi planul } z = h;$$

c) $\iiint\limits_{\Omega} \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}}dxdydz$, unde Ω este domeniul mărginit de elipsoidul $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Soluţii. a) Este convenabil să folosim transformarea:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta\cos\varphi \\ z = r\sin\theta\sin\varphi \end{cases}$$

Noile variabile de integrare sunt r, θ, φ, iar pentru a determina domeniul Ω^* (domeniul transformat), înlocuim x(r, θ, φ), y(r, θ, φ), z (r, θ, φ) în inecuaţiile ce definesc domeniul Ω.



Din $x^2 + y^2 + z^2 \le 4$ rezultă $r^2 \le 4$, deci $r \in [0, 2]$.

Din $y^2 + z^2 \le x^2$ deducem că $r^2\sin^2\theta \le r^2\cos^2\theta$, adică $\sin^2\theta \le \cos^2\theta$, ceea ce este echivalent cu $\sin^2\theta \le \frac{1}{2}$.

(1)

Din $x \ge 0$ rezultă $r\cos\theta \ge 0$, adică $\cos\theta \ge 0$, de unde $\theta \in \left[0, \frac{\pi}{2}\right]$. (2)

Din (1) şi (2) avem $\theta \in \left[0, \frac{\pi}{4}\right]$.

Deci $\Omega^* = \left\{(r,\theta,\varphi) \mid r \in [0,2], \theta \in \left[0,\frac{\pi}{4}\right], \varphi \in [0,2\pi]\right\}$.

Jacobianul transformării este:

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \end{vmatrix} = r^2\sin\theta.$$

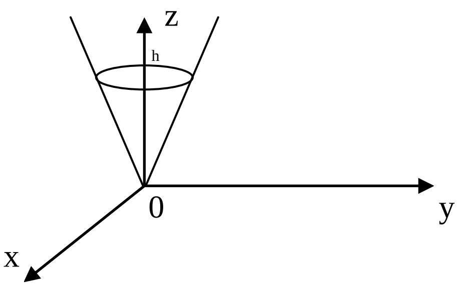
Integral de calculat devine:

$$\int_0^2\int_0^{\pi/4}\left(\int_0^{2\pi} r^4 \sin\theta d\varphi\right)d\theta dr = 2\pi\int_0^2\left(\int_0^{\pi/4} r^4 \sin\theta d\theta\right)dr =$$

$$= 2\pi\int_0^2\left[r^4(-\cos\theta)\ \Big|_{\theta=0}^{\theta=\pi/4}\right]dr = \pi(2-\sqrt{2})\int_0^2 r^4 dr =$$

$$= \pi(2 - \sqrt{2}) \cdot \frac{r^5}{5}\Bigg|_{r=0}^{r=2} = \frac{2^5\cdot\pi\cdot(2-\sqrt{2})}{5}.$$

b) Domeniul pe care se face integrarea este:



Este convenabil să folosim coordonatele cilindrice:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

Avem z $\in$ [0, h], θ $\in$ [0, 2π] iar $z^2 \geq \dfrac{h^2}{R^2}(x^2 + y^2)$ ne dă r $\in$ $\left[0, \dfrac{zR}{h}\right]$.

Aşadar, $\Omega^* = \{(r,\theta,z)\mid 0\leq r\leq \frac{zR}{h}, \theta\in[0,2\pi], z\in[0,h]\}$.

Jacobianul este:

$$\frac{D(x,y,z)}{D(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

Integrala devine:

$$\int_0^h\left[\int_0^{zR/h}\left[\int_0^{2\pi} z r d\theta\right]dr\right]dz = 2\pi\int_0^h\left[\int_0^{zR/h} z r dr\right]dz = 2\pi\int_0^h z\frac{r^2}{2}\Bigg|_{r=0}^{r=zr?h} dz =$$

$$= 2\pi\int_0^h\left(\frac{z}{2}\cdot\frac{z^2R^2}{h^2}\right)dz = \pi\int_0^h\frac{z^3R^2}{h^2}dz = \frac{\pi R^2}{h^2}\cdot\int_0^h z^3 dz = \frac{\pi R^2 h^2}{4}$$

d)Vom folosi coordonatele sferice generalizate, adică:

$$\begin{cases} x = \arcsin\theta\cos\varphi \\ y = br\sin\theta\sin\varphi \\ z = cr\cos\theta \end{cases}$$

Avem $\Omega^* = \{(r, \theta, \varphi) \mid r\in[0, 1], \theta\in[0, \pi], \varphi\in[0, 2\pi]\}$ iar

$$\frac{D(x,y,z)}{D(r,\theta,\varphi)} = abcr^2 \sin\theta.$$

Integrala devine:

$$\int_0^1\int_0^\pi\int_0^{2\pi} abcr^2\sin\theta\sqrt{1-r^2}\,d\varphi d\theta dr = 2\pi abc\int_0^1\int_0^\pi r^2\sqrt{1-r^2}\sin\theta d\theta dr = 2\pi abc$$
$$\int_0^1\left[r^2\sqrt{1-r^2}\cdot(-\cos\theta)\Big|_{\theta=0}^{\theta=\pi}\right]dr = 4\pi abc\int_0^1 r^2\sqrt{1-r^2}\,dr.$$

Pentru calculul acestei din urmă integrale facem schimbarea de variabilă
r =sin t .

Deci $\int_0^1 r^2\sqrt{1-r^2}\,dr = \int_0^{\pi/2}\sin^2 t\cdot\sqrt{1-\sin^2 t}\cos t dt = \int_0^{\pi/2}\sin^2 t\cos^2 t dt =$

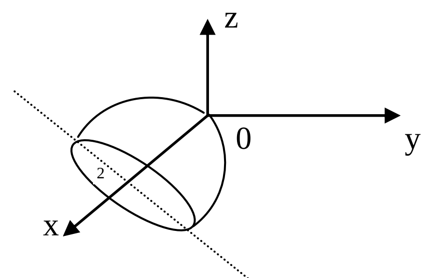
$$= \frac{1}{4}\int_0^{\pi/2}\sin^2 2t dt = \frac{1}{4}\int_0^{\pi/2}\frac{1-\cos 4t}{2}dt = \frac{1}{8}\int_0^{\pi/2}(1-\cos 4t)dt =$$

$$= \frac{1}{8}\left(t-\frac{\sin 4t}{4}\right)\Bigg|_0^{\pi/2} = \frac{1}{8}\left(\frac{\pi}{2}-0\right) = \frac{\pi}{16}.$$

$$\text{Deci: } \iiint_\Omega \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}}\,dxdydz$$
$$= 4\pi abc\cdot\int_0^1 r^2\sqrt{1-r^2}\,dr = 4\pi abc\frac{\pi}{16} = \frac{\pi^2 abc}{4}.$$

Exerciţiul 9.2.12. Să se calculeze volumul corpului mărginit de paraboloidul $x = \frac{y^2}{9}+\frac{z^2}{16}$ şi planul de ecuaţie x = 2.

Soluţie. Corpul Ω al cărui volum trebuie să-l aflăm, este reprezentat în figura următoare:



Vom folosi coordonate cilindrice generalizate:

$$\begin{cases} x = x \\ y = 3r\cos\theta \\ z = 4r\sin\theta \end{cases}, \ x\in[0, 2],\ \theta\in[0, 2\pi].$$

Din x $\geq \dfrac{y^2}{9} + \dfrac{z^2}{16}$ rezultă că x ≥ r^2, deci $0 \leq r \leq \sqrt{x}$.

Aşadar, $\Omega^* = \{(r, \theta, x) \mid 0 \leq r \leq \sqrt{x}, \theta \in [0, 2\pi], x \in [0, 2]\}$.

Jacobianul transformării este : $\dfrac{D(x,y,z)}{D(r,\theta,x)}$ = 12r.

Volumul este:

$$\mathrm{Vol}(\Omega) = \iiint_{\Omega} dxdydz = \iiint_{\Omega} \frac{D(x,y,z)}{D(r,\theta,x)} d\theta dr dx = 12 \iiint_{\Omega^*} r d\theta dr dx =$$

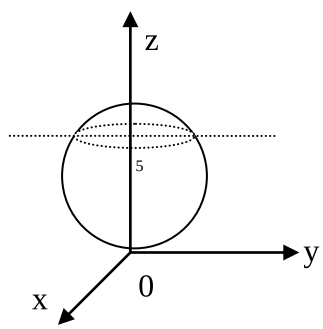
$$= 24\pi \int_0^2 \left(\int_0^{\sqrt{x}} r dr \right) dx = 24\pi \int_0^2 \left[\left. \frac{r^2}{2} \right|_{r=0}^{r=\sqrt{x}} \right] dx = 12\pi \int_0^2 x dx =$$

$$= 12\pi \cdot \left. \frac{x^2}{2} \right|_0^2 = 24\pi$$

Exerciţiul 9.2.13. Să se calculeze masa corpului Ω, mărginit de sfera $x^2 + y^2 + z^2 = 10z$, ştiind că densitatea în fiecare punct este:

$$\rho(x, y, z) = \frac{1}{x^2 + y^2 + z^2} .$$

Soluţie. Se aplică formula $M = \iiint_{\Omega} \rho(x,y,z) dxdydz$.



Avem $z \in [0, 10]$ şi (D_z): $x^2 + y^2 \leq 10z - z^2$.

Deci $M = \displaystyle\int_0^{10} \left[\iint_{D_z} \rho(x,y,z) dxdy \right] dz$.

Pentru calculul integralei duble folosim coordonatele polare. Deci $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$, iar $D_z^* = \{(r, \theta) \mid 0 \leq r \leq \sqrt{10z - z^2}, \theta \in [0, 2\pi]\}$

Avem, aşadar :

$$\iint_{D_z} \rho(x,y,z) dxdy = \int_0^{\sqrt{10z-z^2}} \left[\int_0^{2\pi} \frac{1}{r^2+z^2} r d\theta \right] dr = 2\pi \int_0^{\sqrt{10z-z^2}} \frac{1}{r^2+z^2} dr = \pi \int_0^{\sqrt{10z-z^2}} \frac{(r^2+z^2)'}{r^2+z^2} dr = \pi \cdot \ln\left(r^2+z^2\right)\Big|_{r=0}^{r=\sqrt{10z-z^2}} =$$

$= \pi \ln(10z) - \pi \ln(z^2)$.

Deci, M = $\int_0^{10} [\pi \ln(10z) - \pi \ln(z^2)]dz = \pi \int_0^{10} \ln\left(\frac{10}{z}\right)dz$ =

$= \pi \int_0^{10} \ln 10 dz - \pi \int_0^{10} \ln z dz = \pi \cdot 10 \cdot \ln 10 - \pi \int_0^{10} (z)' \ln z dz =$

$= 10\pi \ln 10 - 10\pi \ln 10 + \pi \int_0^{10} dz = \pi \cdot z\Big|_0^{10} = 10\pi$.

Exercițiul 9.2.14. Să se determine coordonatele centrului de greutate ale segmentului cilindric omogen:

$$\Omega = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \le 9,\ 0 \le z \le 2y\}$$

Soluție. Corpul fiind omogen, funcția ρ este constantă.

Deci, $x_G = \frac{1}{v(\Omega)} \cdot \iiint_\Omega x dxdydz$; $y_G = \frac{1}{v(\Omega)} \cdot \iiint_\Omega y dxdydz$;

$z_G = \frac{1}{v(\Omega)} \cdot \iiint_\Omega z dxdydz$.

Notând $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le 9,\ y \ge 0\}$. Avem:

- $v(\Omega) = \iiint_\Omega dxdydz = \iint_D \left(\int_0^{2y} dz\right) dxdy = \iint_D 2y dxdy$ =

  $= 2\iint_D y dxdy = 2\iint_{D^*} r^2 \sin\theta dr d\theta = 2\int_0^3 \left(\int_0^\pi r^2 \sin\theta d\theta\right) dr$

  $= 2\int_0^3 \left[r^2 (-\cos\theta)\Big|_{\theta=0}^{\theta=\pi}\right] dr = 4\int_0^3 r^2 dr = 4\frac{r^3}{3}\Big|_0^3 = 36$

- $\iiint_\Omega x dxdydz = \iint_D \left(\int_0^{2y} x dz\right) dxdy = \iint_D xz\Big|_{z=0}^{z=2y} dxdy$ =

  $= \iint_D 2xy dxdy = 2\iint_{D^*} r^3 \sin\theta \cos\theta dr d\theta =$

  $= \int_0^3 r^3 dr \cdot \int_0^\pi \sin 2\theta d\theta = \frac{r^4}{4}\Big|_0^3 \cdot \left(-\frac{\cos 2\theta}{2}\right)\Big|_0^\pi = 0$

- $\iiint_\Omega y dxdydz = \iint_D \left(\int_0^{2y} y dz\right) dxdy = \iint_D yz\Big|_{z=0}^{z=2y} dxdy$ =

  $= \iint_D 2y^2 dxdy = 2\iint_{D^*} r^3 \sin^2\theta dr d\theta =$

  $= \frac{r^4}{4}\Big|_0^3 \cdot \int_0^\pi (1 - \cos 2\theta) d\theta = \frac{3^4}{4}\left(\pi - \frac{\sin 2\theta}{2}\Big|_0^\pi\right) = 3^4 \frac{\pi}{4}$

- $\iiint_\Omega z dxdydz = \iint_D \left(\int_0^{2y} z dz\right) dxdy = \iint_D \frac{z^2}{2}\Big|_{z=0}^{z=2y} dxdy$ =

$$= \frac{1}{2}\iint_D 4y^2 dxdy = 2\iint_D y^2 dxdy = 3^4\,\frac{\pi}{4}.$$

Rezultă $x_G = 0$; $y_G = z_G = \frac{1}{36}\cdot 3^4\,\frac{\pi}{4} = \frac{9\pi}{16}$.

Exercițiul 9.2.15. Să se calculeze momentele de inerție în raport cu planele de coordonate ale corpului material omogen, limitat de suprafețele $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}$ și $z = c$.

Soluție. $I_{xOy} = \iiint_\Omega z^2 dxdydz$ (corpul fiind omogen, considerăm densitatea egală cu unitatea).

Trecem la coordonate cilindrice generalizate:

$$\begin{cases} x = ar\cos\theta \\ y = br\sin\theta \\ z = z \end{cases},\ z\in[0, c],\ \theta\in[0, 2\pi]$$

Din $\frac{x^2}{a^2}+\frac{y^2}{b^2}\le\frac{z^2}{c^2}$, obținem $r^2 \le \frac{z^2}{c^2}$, de unde $0 \le r \le \frac{z}{c}$.

Deci $\Omega^* = \{(r, \theta, z) \mid 0 \le r \le \frac{z}{c},\ \theta\in[0, 2\pi],\ z\in[0, c]\}$, iar jacobianul transformării este $\frac{D(x,y,z)}{D(r,\theta,z)} = abr$.

- $I_{xOy} = ab\iiint_{\Omega^*} z^2 r\,d\theta drdz = 2\pi ab\int_0^c\left(\int_0^{z/c} z^2 r dr\right)dz =$

$$= 2\pi ab\int_0^c\left[z^2\,\frac{r^2}{2}\Big|_{r=0}^{r=\frac{z}{c}}\right]dz = \frac{\pi ab}{c^2}\int_0^c z^4 dz = \frac{\pi ab}{c^2}\cdot\frac{z^5}{5}\Big|_0^c = \frac{\pi abc^3}{5}$$

- $I_{yOz} = \iiint_\Omega x^2 dxdydz = a^3 b\iiint_{\Omega^*} r^3\cos^2\theta\,d\theta drdz =$

$$= a^3 b\int_0^c\int_0^{2\pi}\left[\int_0^{z/c} r^3\cos^2\theta\,dr\right]d\theta dz = \frac{a^3 b}{4}\int_0^c\int_0^{2\pi}\frac{z^4}{c^4}\cos^2\theta\,d\theta dz =$$

$$= \frac{a^3 b}{4c^4}\int_0^c\left(\int_0^{2\pi}\frac{z^4(1+\cos 2\theta)}{2}\,d\theta\right)dz = \frac{a^3 b}{8c^4}\int_0^c z^4\cdot 2\pi dz\ +$$

$$+\ \frac{a^3 b}{8c^4}\int_0^c\left[z^4\cdot\frac{\sin 2\theta}{2}\Big|_{\theta=0}^{\theta=2\pi}\right]dz = \frac{\pi a^3 b}{4c^4}\cdot\frac{z^5}{5}\Big|_{z=0}^{z=c} = \frac{\pi a^3 bc}{20}$$

- $I_{xOz} = \iiint_\Omega y^2 dxdydz = ab^3\iiint_{\Omega^*} r^3\sin^2\theta\,d\theta drdz =$

$$= ab^3\int_0^c\int_0^{2\pi}\frac{z^4}{4c^4}\sin^2\theta\,d\theta dz = \frac{ab^3}{4c^4}\int_0^c\int_0^{2\pi}\frac{z^4(1-\cos 2\theta)}{2}d\theta dz =$$

$$= \frac{ab^3}{8c^4}\int_0^c z^4\cdot 2\pi dz - \frac{ab^3}{8c^4}\int_0^c\left[z^4\cdot\frac{\sin 2\theta}{2}\Big|_{\theta=0}^{\theta=2\pi}\right]dz = \frac{\pi ab^3}{4c^4}\cdot\frac{z^5}{5}\Big|_{z=0}^{z=c} = \frac{\pi ab^3 c}{20}$$