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# MATHEMATICAL ANALYSIS

## VOL. II

## INTEGRAL CALCULUS

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## § V.1. DEFINITE INTEGRALS WITH PARAMETERS

We consider that the integral calculus for the functions of one real variable is known. Here we include the indefinite integrals (also called primitives or anti-derivatives) as well as the definite integrals. Similarly, we consider that the basic methods of calculating (exactly and approximately) integrals are known.

The purpose of this paragraph is to study an extension of the notion of definite integral in the sense that beyond the variable of integration there exists another variable also called parameter.

**1.1. <u>Definition</u>.** Let us consider an interval $A \subseteq \mathbb{R}$, $I = [a, b] \subset \mathbb{R}$ and $f : A \times I \to \mathbb{R}$. If for each $x \in A$ ($x$ is called *parameter*), function $t \mapsto f(x, t)$ is integrable on $[a, b]$, then we say that $F : A \to \mathbb{R}$, defined by

$$F(x) = \int_a^b f(x, t)dt$$

is a *definite integral with parameter* (between *fixed limits a* and *b*).

More generally, if instead of $a, b$ we consider two functions $\varphi, \psi : A \to [a, b]$ such that $\varphi(x) \le \psi(x)$ for all $x \in A$, and the function $t \mapsto f(x, t)$ is integrable on the interval $[\varphi(x), \psi(x)]$ for each $x \in A$, then the function

$$G(x) = \int_{\varphi(x)}^{\psi(x)} f(x, t)dt$$

is called *definite integral with parameter* $x$ (between *variable limits*).

The integrals with variable limits may be reduced to integrals with constant limits by changing the variable of integration:

**1.2. <u>Lemma</u>.** In the conditions of the above definition, we have:

$$G(x) = [\psi(x) - \varphi(x)]\int_0^1 f(x, \varphi(x) + \theta[\psi(x) - \varphi(x)])d\,\theta\,.$$

<u>Proof.</u> In the integral $G(x)$ we make the change $t = \varphi(x) + \theta\,[\psi(x) - \varphi(x)]$, for which $\frac{dt}{d\theta} = \psi(x) - \varphi(x)$. ◇

Relative to $F$ and $G$ we'll study the properties concerning continuity, derivability and integrability in respect to the parameter.

**1.3. <u>Theorem</u>.** If $f : A \times I \to \mathbb{R}$ is continuous on $A \times I$, then $F : A \to \mathbb{R}$ is continuous on $A$.

Proof. If $x_0 \in A$, then either $x_0 \in \mathring{A}$, or $x_0$ is an end-point of $A$. In any case there exists $\eta > 0$ such that

$$K_\eta = \{(x, t) \in \mathbb{R}^2 : | x - x_0| \le \eta \ , x \in A, t \in [a, b]\}$$

is a compact part of $A \times I$. Since $f$ is continuous on $A \times I$, it will be uniformly continuous on $K_\eta$ , i.e. for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$| f(x', t') - f(x'', t'') | < \frac{\varepsilon}{2(b-a)}$$

whenever $(x', t'), (x'', t'') \in K_\eta$ and $d((x', t'), (x'', t'')) < \delta$.

Consequently, for all $x \in A$ for which $| x - x_0 | < min \{ \eta , \delta \}$ we have

$$| F(x) - F(x_0) | \le \int_a^b | f(x, t) - f(x_0, t)|dt \le \frac{\varepsilon}{2(b-a)}(b - a) < \varepsilon,$$

which means that $F$ is continuous at $x_0$ . ◇

**1.4. Corollary.** If the function $f : A \times I \to \mathbb{R}$ is continuous on $A \times I$, and $\varphi, \psi : A \to [a, b]$ are continuous on $A$, then $G : A \to \mathbb{R}$ is continuous on $A$.

Proof. Function $g : A \times [0, 1] \to \mathbb{R}$, defined by

$$g(x, \theta) = f(x, \varphi(x) + \theta[\psi(x) - \varphi(x)]),$$

which was used in lemma 1.2, is continuous on $A \times [0, 1]$, hence we can apply theorem 1.3 and lemma 1.2. ◇

**1.5. Theorem.** Let $A \subseteq \mathbb{R}$ be an arbitrary interval, $I = [a, b] \subset \mathbb{R}$, and let us note $f : A \times I \to \mathbb{R}$. If $f$ is continuous on $A \times I$, and it has a continuous partial derivative $\frac{\partial f}{\partial x}$, then $F \in C_{\mathbb{R}}^1(A)$, and $F'(x) = \int_a^b \frac{\partial f}{\partial x}(x, t)dt.$

Proof. We have to show that at each $x_0 \in A$, there exists

$$\lim_{x \to x_0} \frac{F(x) - F(x_0)}{x - x_0} = \int_a^b \frac{\partial f}{\partial x}(x_0, t)dt \ .$$

For this purpose we consider the following helpful function

$$h(x, t) = \begin{cases} \dfrac{f(x,t) - f(x_0,t)}{x - x_0} & \text{if } x \ne x_0 \\ \dfrac{\partial f}{\partial x}(x_0, t) & \text{if } x = x_0 \end{cases}$$

On the hypothesis it is clear that $h$ is continuous on $A \times I$, hence we can use theorem 1.3 for the function

$$H(x) = \int_a^b h(x, t)dt = \int_a^b \frac{f(x,t) - f(x_0,t)}{x - x_0}dt = \frac{F(x) - F(x_0)}{x - x_0}.$$

On this way, the equality $H(x_0) = \lim\limits_{x \to x_0} H(x)$ shows that $F$ is derivable at $x_0$, and

$$F'(x_0) = \int_a^b \frac{\partial f}{\partial x}(x_0, t)dt.$$

The continuity of $F'$ is a consequence of the continuity of $\frac{\partial f}{\partial x}$, by virtue of the same theorem 1.3. ◇

**1.6. Corollary.** If, in addition to the hypothesis of the above theorem, we have $\varphi, \psi \in C_{\mathbb{R}}^1(A)$, then $G \in C_{\mathbb{R}}^1(A)$ and the equality

$$G'(x) = \int_{\varphi(x)}^{\psi(x)} \frac{\partial f}{\partial x}(x, t)dt + f(x, \psi(x))\, \psi'(x) - f(x, \varphi(x))\, \varphi'(x)$$

holds at any $x \in A$.

Proof. Let us consider a new function $L : A \times I \times I \to \mathbb{R}$, expressed by $L(x,u,v) = \int_u^v f(x, t)dt$ . According to the above theorem, for fixed $u$ and $v$ we have $\frac{\partial L}{\partial x}(x,u,v) = \int_u^v \frac{\partial f}{\partial x}(x,t)dt$. On the other hand, the general properties of a primitive lead to $\frac{\partial L}{\partial u}(x, u, v) = -f(x, u)$ and $\frac{\partial L}{\partial v}(x, u, v) = f(x, v)$. Because all these partial derivatives are continuous, $L$ is differentiable on $A \times I \times I$. Applying the rule of deriving a composite function in the case of $G(x) = L(x, \varphi(x), \psi(x))$, we obtain the announced formula. The continuity of $G'$ follows by using theorem 1.3. ◇

**1.7. Theorem.** If $f : A \times I \to \mathbb{R}$ is continuous on $A \times I$ , then $F : A \to \mathbb{R}$ is integrable on any compact $[\alpha, \beta] \subseteq A$, and

$$\int_\alpha^\beta F(x)dx = \int_a^b \left[ \int_\alpha^\beta f(x,t)dx \right] dt.$$

Proof. According to theorem 1.3, $F$ is continuous on $[\alpha, \beta]$, hence it is also integrable on this interval. It is well known that the function

$$\Phi(y) = \int_\alpha^y F(x)dx$$

is a primitive of $F$ on $[\alpha, \beta]$. We will show that

$$\Phi(y) = \int_a^b \left[ \int_\alpha^y f(x,t)dx \right] dt .$$

For this purpose let us note $U(y, t) = \int_\alpha^y f(x, t)dx$ and $\Psi(y) = \int_a^b U(y, t)dt$.

Then, $\frac{\partial U}{\partial y}(y, t) = f(y, t)$, hence according to theorem 1.5, we have $\Psi'(y) = \int_a^b f(y,t)dt$. Consequently, the equalities $\Psi'(y) = F(y) = \Phi'(y)$ hold at any $y \in [\alpha, \beta]$, hence $\Phi(y) - \Psi(y) = c$, where c is *a* constant. Because $\Phi(\alpha) = \Psi(\alpha) = 0$, we obtain $c = 0$, i.e. $\Phi = \Psi$. In particular, $\Phi(\beta) = \Psi(\beta)$ express the required equality. ◇

**1.8. <u>Corollary</u>.** If, in addition to the conditions in the above theorem, the functions $\varphi, \psi : A \to [a, b]$ are continuous on $A$, then

$$\int_\alpha^\beta G(x)dx = \int_0^1 \left[ \int_\alpha^\beta g(x,\theta)dx \right] d\theta$$

where $g(x, \theta) = f(x, \varphi(x) + \theta[\psi(x) - \varphi(x)]) [\psi(x) - \varphi(x)]$ (as in corollary 4).

<u>Proof</u>. According to Lemma 1.2, we have $G(x) = \int_0^1 g(x,\theta)d\theta$, so it remains to use theorem 1.7. ◇

**1.9. <u>Remark</u>.** The formulas established in the above theorems and their corollaries (especially that which refers to derivation and integration) are frequently useful in practice for calculating integrals (see the problems at the end of the paragraph). In particular, theorem 1.7 gives the conditions on which we can change the order in an iterated integral, i.e.

$$\int_\alpha^\beta \left[ \int_a^b f(x,t)dt \right] dx = \int_a^b \left[ \int_\alpha^\beta f(x,t)dx \right] dt .$$

## PROBLEMS § V.1

**1.** Calculate $\int_0^{\pi/2} \ln(x^2 - \sin^2 t)dt$ , where $x > 1$.

Hint. Denoting the integral by $F(x)$, we obtain $F'(x) = \int_0^{\pi/2} \frac{2x}{x^2 - \sin^2 t} dt$.

Using the substitution $tg\,\frac{t}{2} = u$, we obtain $F'(x) = \frac{\pi}{\sqrt{x^2 - 1}}$, and so

$F(x) = \pi\, ln(x + \sqrt{x^2 - 1}) + c$. In order to find $c$, we write

$$c = F(x) - \pi\, ln(x + \sqrt{x^2 - 1}) =$$

$$= \int_0^{\pi/2} \left[ \ln x^2 + \ln\left(1 - \frac{\sin^2 t}{x^2}\right)\right] dt - \pi ln(x + \sqrt{x^2 - 1}) =$$

$$= \int_0^{\pi/2} \ln\left(1 - \frac{\sin^2 t}{x^2}\right) dt - \pi \ln \frac{x + \sqrt{x^2 - 1}}{x}.$$

Taking here $x \to \infty$, it follows $c = -\pi ln\, 2$.

**2.** Calculate $I = \int_0^1 f(x)dx$, where $f : [0, 1] \to \mathbb{R}$ has the values

$$f(x) = \begin{cases} \dfrac{x^\beta - x^\alpha}{\ln x} & \text{if } x \in (0,1),\ 0 < \alpha < \beta \\ 0 & \text{if } x = 0, x = 1 \end{cases}$$

Hint. Notice that $f(x) = \int_\alpha^\beta x^t dt$ at any $x \in [0, 1)$, and at the end point 1, there exists $\lim_{x \to 1} f(x) = \beta - \alpha$, so only at this point $f$ differs form a continuous function on $[0, 1]$. Consequently $I = \int_0^1 [\int_\alpha^\beta x^t dt\,]dx = \int_\alpha^\beta \left[\int_0^1 x^t dx\right] dt = \ln \frac{\beta + 1}{\alpha + 1}$.

**3.** Calculate $\lim_{x \to 0} \frac{\int_0^{\sin x} e^{xt^2} dt}{\int_0^{tgx} e^{-xt^2} dt}$ .

Hint. This is a $\frac{0}{0}$ indetermination; in order to use L'Hospital rule we need the derivatives relative to $x$, which is a parameter in the upper limits of integrals, so the limit reduces to

$$\lim_{x\to 0}\frac{e^{x\sin^2 x}\cos x+\int\limits_0^{\sin x} t^2 e^{xt^2}\,dt}{e^{-xtg^2 x}\cos^{-2} x+\int\limits_0^{tgx}(-t^2)e^{-xt^2}\,dt}=1\,.$$

**4.** Calculate $I=\int\limits_0^{\pi}\frac{dx}{a+b\cos x}$, where $0<|\,b\,|<a$, and deduce the values of

$I=\int\limits_0^{\pi}\frac{dx}{a+b\cos x}$, $K=\int\limits_0^{\pi}\frac{\cos x}{(a+b\cos x)^2}dx$ and $L=\int\limits_0^{\pi}\ln(a+b\cos x)dx$.

Hint. The substitution $tg\,\frac{x}{2}=t$ is not possible in $I$ because $[0, \pi)$ is carried into $[0, \infty)$. Since the integral is continuous on $\mathbb{R}$, we have

$$I=\lim_{l\to\pi}\int\limits_0^{l}\frac{dx}{a+b\cos x},$$

and this last integral can be calculated using the mentioned substitution. More exactly,

$$\int\limits_0^{l}\frac{dx}{a+b\cos x}=2\int\limits_0^{tg\frac{l}{2}}\frac{dt}{a+b+t^2(a-b)}=\frac{2}{\sqrt{a^2-b^2}}\,arctg\left(\sqrt{\frac{a-b}{a+b}}\,tg\,\frac{l}{2}\right)$$

hence $I=\frac{\pi}{\sqrt{a^2-b^2}}$. To obtain $K$, we derive $I$ relative to $b$. Finally, $\frac{\partial L}{\partial a}=I$.

**5.** Calculate $I=\int\limits_0^{1}\frac{arctgx}{x\sqrt{1-x^2}}dx$ by deriving $I(y)=\int\limits_0^{1}\frac{arctgxy}{x\sqrt{1-x^2}}dx$, $y\geq 0$.

Hint. Substitution $x=\cos\theta$ gives

$$I'(y)=\int\limits_0^{1}\frac{dx}{(1+x^2y^2)\sqrt{1-x^2}}=\int\limits_0^{\pi/2}\frac{d\theta}{1+y^2\cos^2\theta}.$$

Because the substitution $tg\ \theta=t$ carries $[0, \frac{\pi}{2})$ into $[0, \infty)$, and the substitution $tg\,\frac{\theta}{2}=t$ leads to a complicated calculation, we consider

$$I'(y) = \lim_{l \to \frac{\pi}{2}} \int_0^l \frac{d\theta}{1 + y^2 \cos^2 \theta}.$$

If we replace $tg\,\theta = t$ in this last integral, then we obtain

$$\int_0^l \frac{d\theta}{1 + y^2 \cos^2 \theta} = \int_0^{tgl} \frac{dt}{1 + y^2 + t^2} = \frac{1}{\sqrt{1 + y^2}} arctg \frac{tgl}{\sqrt{1 + y^2}}.$$

Consequently $I'(y) = \dfrac{\pi}{2} \dfrac{1}{\sqrt{1 + y^2}}$, hence $I(y) = \dfrac{\pi}{2} ln(y + \sqrt{1 + y^2}) + c$.

Because $I(0) = 0$ it follows that $c = 0$, hence $I = I(1) = \dfrac{\pi}{2} ln(1 + \sqrt{2})$.

**6.** Calculate $I = \displaystyle\int_0^1 \frac{arctgx}{x\sqrt{1 - x^2}} dx$ using the formula $\dfrac{arctgx}{x} = \displaystyle\int_0^1 \frac{dy}{1 + x^2 y^2}$.

Hint. Changing the order of integration we obtain

$$I = \int_0^1 \frac{1}{\sqrt{1 - x^2}} \left( \int_0^1 \frac{dy}{1 + x^2 y^2} \right) dx = \int_0^1 \left[ \int_0^1 \frac{dx}{(1 + x^2 y^2)\sqrt{1 - x^2}} \right] dy$$

so the problem reduces to $I'(y)$ from problem 5.

**7.** Calculate $K = \displaystyle\int_0^{\frac{\pi}{2}} \ln \frac{a + b \sin x}{a - b \sin x} \cdot \frac{dx}{\sin x}$, $a > b > 0$.

Hint. Using the formula $\dfrac{1}{\sin x} \cdot \ln \dfrac{a + b \sin x}{a - b \sin x} = 2ab \displaystyle\int_0^1 \frac{dy}{a^2 - b^2 y^2 \sin^2 x}$ we obtain

$$K = 2ab \int_0^{\frac{\pi}{2}} \left[ \int_0^1 \frac{dy}{a^2 - b^2 y^2 \sin^2 x} \right] dx = 2ab \int_0^1 \left[ \int_0^{\frac{\pi}{2}} \frac{dx}{a^2 - b^2 y^2 \sin^2 x} \right] dy.$$

Since $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{a^2 - b^2 y^2 \sin^2 x} = \frac{\pi}{2a\sqrt{a^2 - b^2 y^2}}$ it follows that

$$K = \pi b \int_0^1 \frac{dy}{\sqrt{a^2 - b^2 y^2}} = \pi \arcsin \frac{b}{a}.$$

**8.** Show that $I_{n+1}(a) = \dfrac{-1}{2na} I'_n(a)$, where $I_n(a) = \displaystyle\int_0^1 \frac{dx}{(x^2+a^2)^n}$, $n \in \mathbb{N}^*$, $a \neq 0$. Using this result, calculate $\displaystyle\int_0^1 \frac{dx}{(1+x^2)^3}$.

Hint. Derive $I_n(a)$ relative to $a$ .

**9.** Use Theorem 1.7 to evaluate $I = \displaystyle\int_0^1 f(x)dx$, where

$$f(x) = \begin{cases} \dfrac{x^\beta - x^\alpha}{\ln x}\sin(\ln x) & , \textit{if}\ \ x \neq 0\ \textit{and}\ \ x \neq 1 \\ 0 & , \textit{if}\ \ x = 0\ \textit{or}\ \ x = 1 \end{cases}$$

and $\alpha > 0$, $\beta > 0$.

Hint. Introduce a parameter $t$ and remark that

$$I = \int_0^1 \left[ \int_\alpha^\beta x^t\, dt \right] \cdot \sin(\ln x)\, dx\ .$$

Change the order of integration to obtain

$$I = \int_\alpha^\beta \left[ \int_0^1 x^t \sin(\ln x) dx \right] dt\ = -\int_\alpha^\beta \frac{dt}{1+(t+1)^2}\ .$$

The result is $I = arctg \dfrac{\alpha - \beta}{1+(\alpha+1)(\beta+1)}$ .

## §V.2. IMPROPER INTEGRALS

In the construction of the definite integral, noted $\int_a^b f(t)\,dt$, we have used two conditions which allow us to write the integral sums, namely:

(i) $a$ and $b$ are finite (i.e. different from $\pm\infty$);

(ii) $f$ is bounded on $[a, b]$, where it is defined.

There are still many practical problems, which lead to integrals of functions not satisfying these conditions. Even definite integrals reduce sometimes to such "more general" integrals, as for example when changing the variables by $tg\,\frac{x}{2} = t$, the interval $[0, \pi]$ is carried into $[0, \infty]$.

The aim of this paragraph is to extend the notion of integral in the case when these conditions are no longer satisfied.

**2.1. Definition.** *The case when* $b = \infty$. If $f : [a, \infty) \to \mathbb{R}$ is integrable on $[a, \beta]$ for all $\beta > a$, and there exists $L = \lim\limits_{\beta\to\infty} \int\limits_a^\beta f(t)dt$, then we may say that $f$ is *improperly integrable* on $[a, \infty)$, and $L$ is the *improper integral* of $f$ on $[a, \infty)$. In this case we note $\int\limits_a^\infty f(t)dt = \lim\limits_{\beta\to\infty} \int\limits_a^\beta f(t)dt$, and we say that the improper integral is *convergent.*

Similarly we discuss the case when $a = -\infty$.

*The case when f is unbounded at b*. Let $f : [a, b) \to \mathbb{R}$ be unbounded in the neighborhood of $b$, in the sense that for arbitrary $\delta > 0$ and $M > 0$ there exists $t \in (b - \delta, b)$ such that $f(t) > M$. If $f$ is integrable on $[a, \beta]$ for all $a < \beta < b$, and there exists $L = \lim\limits_{\beta\to b} \int\limits_a^\beta f(t)dt$, then we say that $f$ is *improperly integrable on* $[a, b)$, and $L$ is called *improper integral* of $f$ on $[a, b)$. If $L$ exists, we note $\int\limits_a^b f(t)dt = \lim\limits_{\beta\to b} \int\limits_a^\beta f(t)dt$, and we say that the improper integral is *convergent.*

We similarly treat the functions which are unbounded at $a$ .

**2.2. Remarks.** a) In practice we often deal with combinations of the above simple situations, as for example

$$\int\limits_{-\infty}^{+\infty} f(t)dt \overset{not.}{=} \int\limits_{\mathbb{R}} f(t)dt = \lim_{\substack{\alpha\to-\infty \\ \beta\to+\infty}} \int\limits_\alpha^\beta f(t)dt,$$

$$\int_a^b f(t)dt = \lim_{\substack{\alpha\to a \\ \beta\to b}} \int_\alpha^\beta f(t)dt, \text{ where } a < \alpha < \beta < b.$$

The integral $\int_a^b f(t)dt$ can be improper because $f$ is unbounded at some point $c \in (a, b)$, in which case we define

$$\int_a^b f(t)dt = \lim_{\substack{\alpha\to c \\ \alpha<c}} \int_a^\alpha f(t)dt + \lim_{\substack{\beta\to c \\ \beta>c}} \int_\beta^b f(t)dt.$$

b) From the geometrical point of view, considering improper integrals may be interpreted as measuring areas of unbounded subsets of the plane. The existence of the above considered limits shows that we can speak of the area of an unbounded set, at least for sub-graphs of some real functions.
c) In spite of the diversity of types of improper integrals, there is a simple, but essential common feature, namely that the integration is realized *on non-compact sets*. In fact, a compact set in $\mathbb{R}$ is bounded and closed, hence $[a, \infty)$, $(-\infty, b]$, $(-\infty, +\infty)$ are non-compact because they are not bounded, while $[a, b)$, $(a, b]$, etc. are non-compact because of non-closeness. Obviously, other combinations like $(a, \infty)$, $(-\infty, c) \cup (c, b]$, etc. are possible. Because any improper integral is defined by a limiting process, when proving some property of such integrals it is sufficient to consider only one of the possible cases.

**2.3. <u>Examples</u>.** a) The integral $I(\lambda) = \int_1^\infty \frac{dt}{t^\lambda}$ ($\lambda \in \mathbb{R}$) is convergent for $\lambda > 1$, when $I(\lambda) = (\lambda - 1)^{-1}$, and divergent for $\lambda \le 1$. In fact, according to the above definition, $I(\lambda) = \lim_{\beta\to\infty} \int_1^\beta t^{-\lambda} dt$, where

$$\int_1^\beta t^{-\lambda} dt = \begin{cases} \dfrac{-1}{1-\lambda}(1-\beta^{1-\lambda}) & \text{if } \lambda \ne 1 \\ \ln\beta & \text{if } \lambda = 1. \end{cases}$$

Finally, it remains to remember that

$$\lim_{\beta\to\infty} \beta^{1-\lambda} = \begin{cases} 0 & \text{if } \lambda > 1 \\ 1 & \text{if } \lambda = 1 \\ \infty & \text{if } \lambda < 1. \end{cases}$$

b) The integral $I(\mu) = \int_0^1 \frac{dt}{t^\mu}$ $(\mu > 0)$ is convergent for $\mu < 1$, when it equals $I(\mu) = (1 - \mu)^{-1}$, and it is divergent for $\mu \geq 1$.

Figures V.2.1. a), respectively b), suggest how to interpret $I(\lambda)$ and $I(\mu)$ as areas of some sub-graphs (hatched portions).

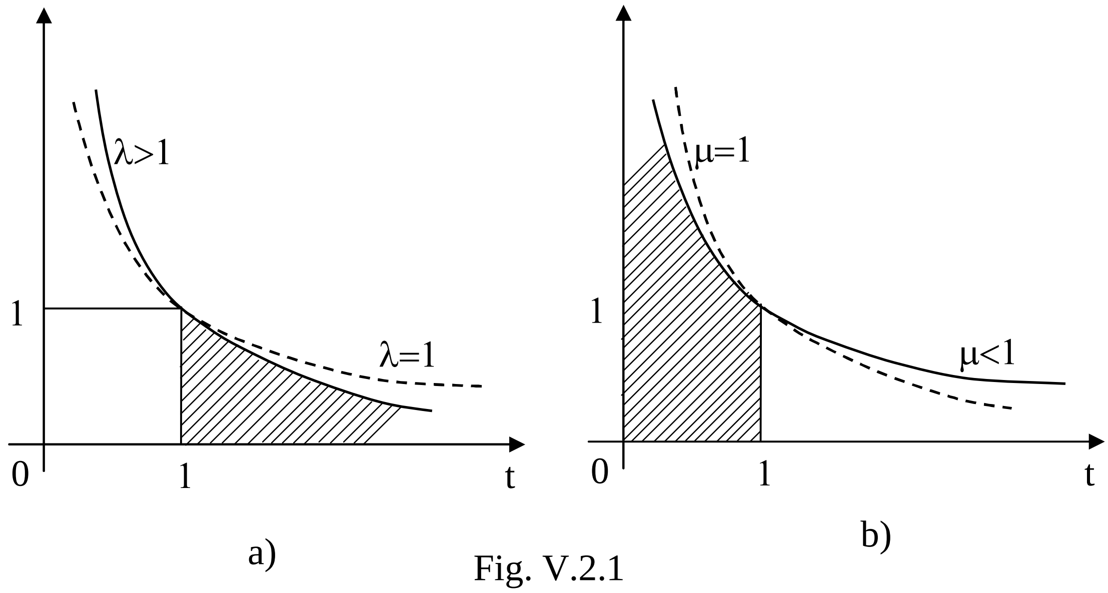



Fig. V.2.1

The usual properties of the definite integrals also hold for improper integrals, namely:

**2.4. <u>Proposition</u>.** a) The improper integral is a linear functional on the space of all improperly integrable functions, i.e. if $f, g : [a, b) \to \mathbb{R}$ are improperly integrable on [a, b), and $\lambda, \mu \in \mathbb{R}$, then $\lambda f + \mu g$ is improperly integrable on [a, b) and we have:

$$\int_a^b (\lambda f + \mu g)(t)dt = \lambda \int_a^b f(t)dt + \mu \int_a^b g(t)dt.$$

b) The improper integral is additive relative to the interval, i.e.

$$\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt.$$

c) The improper integral is dependent on the order of the interval, namely

$$\int_a^b f(t)dt = -\int_b^a f(t)dt.$$

**2.5. <u>Theorem.</u>** (**Leibniz-Newton** formula) Let $f : [a, b) \to \mathbb{R}$ be (properly) integrable on any compact $[a, \beta]$included in [a, b), and $F$ be the primitive of $f$ on [a, b). Then a necessary and sufficient condition for $f$ to be improperly integrable on [a, b) is to exist the finite limit of $F$ at $b$. In this case we have:

$$\int_a^b f(t)dt = \lim_{\beta\to b} F(\beta) - F(a).$$

**2.6. Theorem.** (Integration by parts) If *f, g* satisfy the conditions:

(i) $f, g \in C^1_{\mathbb{R}}([a, b])$

(ii) there exists and is finite $\lim\limits_{\substack{x\to b\\ x<b}} (fg)(x)$

(iii) $\int_a^b f(t)g'(t)dt$ is convergent

then $\int_a^b f'(t)g(t)dt$ is convergent too, and we have

$$\int_a^b f'(t)g(t)dt = \lim_{\substack{x\to b\\ x<b}} (fg)(x) - f(a)g(a) - \int_a^b f(t)g'(t)dt.$$

**2.7. Theorem.** (Changing the variable) Let $f : [a, b) \to \mathbb{R}$ be continuous on [a, b), and let $\varphi : [a', b') \to [a, b)$ be of class $C^1_{\mathbb{R}}([a', b'])$, such that $\varphi(a') = a$ and $\lim\limits_{\substack{\theta\to b'\\ \theta<b'}} \varphi(\theta) = b$. If $\int_a^b f(t)dt$ is convergent, then the integral

$$\int_{a'}^{b'} f(\varphi(\theta))\varphi'(\theta)d\theta$$

is also convergent, and we have

$$\int_{a'}^{b'} f(\varphi(\theta))\varphi'(\theta)d\theta = \int_a^b f(t)dt.$$

The above properties (especially theorems 2.5 – 2.7) are useful in the cases when primitives are available. If the improper integral can't be calculated using the primitives it is still important to study the convergence. For developing such a study we have several *tests of convergence*, as follows:

**2.8. Theorem.** (Cauchy's general test) Let $f : [a, b) \to \mathbb{R}$ be (properly) integrable on any $[a, \beta] \subset [a, b)$. Then $\int_a^b f(t)dt$ is convergent iff for every $\varepsilon > 0$ there exists $\delta > 0$ such that $b', b'' \in (b - \delta, b)$ implies $\left|\int_{b'}^{b''} f(t)dt\right| < \varepsilon$.

Proof. Let $F : [a, b) \to \mathbb{R}$ be defined by $F(x) = \int_a^x f(t)dt$. Then $f$ is improperly integrable on [a, b) if $F$ has a finite limit at $b$, which means that for every $\varepsilon > 0$ we can find $\delta > 0$ such that $b', b'' \in (b - \delta, b)$ implies $|F(b') - F(b'')| < \varepsilon$. It remains to remark that $F(b') - F(b'') = -\int_{b'}^{b''} f(t)dt$. ◇

The above Cauchy's general test is useful in realizing analogies with absolutely convergent series as follows:

**2.9. Definition.** If $f : [a, b) \to \mathbb{R}$, then we say that the integral $\int_a^b f(t)dt$ is *absolutely convergent* iff $\int_a^b | f(t) | dt$ is convergent, i.e. $|f|$ is improperly integrable on [a, b).

**2.10. Remark**. In what concerns the integrability of $f$ and $|f|$, the improper integral differs from the definite integral: while "*f integrable*" in the proper sense implies "$|f|$ *integrable*", this is not valid for improper integrals. In fact, there exist functions, which are improperly integrable without being absolutely integrable. For example, let $f : [0, \infty) \to \mathbb{R}$ be a function of values $f(0) = 1$, and $f(t) = \frac{(-1)^{n-1}}{n}$ if $t \in (n-1, n]$, where $n \in \mathbb{N}^*$. This function is improperly integrable on $[0, \infty)$, and

$$\int_0^\infty f(t)dt = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{n} = \ln 2 ,$$

but it is not absolutely integrable since

$$\int_0^\infty |f(t)|dt = \sum_{n=1}^{\infty} \frac{1}{n} = \infty .$$

The next proposition shows that the opposite implication holds for the improper integrals:

**2.11. Proposition.** Every absolutely convergent integral is convergent.

Proof. Using the Cauchy's general test, the hypothesis means that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any $\beta', \beta'' \in (b - \delta, b)$ we have

$$\int_{\beta'}^{\beta''} |f(t)dt| < \varepsilon.$$

Because $f$ is properly integrable on any compact from [a, b), and

$$\left|\int_{\beta'}^{\beta''} f(t)dt\right| \le \int_{\beta'}^{\beta''} |f(t)dt| = \left| \int_{\beta'}^{\beta''} |f(t)dt| \right|$$

it follows that $f$ is improperly integrable on $[a, b)$. ◇

**2.12. Theorem.** (The comparison test) Let $f, g : [a, b) \to \mathbb{R}$ be such that:

1) $f, g$ are properly integrable on any compact from [a, b)
2) for all $t \in$ [a, b) we have $|f(t)| \le g(t)$
3) $\int_a^b g(t)dt$ is convergent.

Then $\int_a^b f(t)dt$ is absolutely convergent.

Proof. Because $\int_{\beta'}^{\beta''} |f(t)|dt \le \int_{\beta'}^{\beta''} g(t)dt$ holds for all $\beta', \beta'' \in (b-\delta, b)$, $\beta' < \beta''$, we can apply the Cauchy's general test. ◇

**2.13. Remark.** a) Besides its utility in establishing convergence, the above theorem can be used as a divergence test. In particular, if $0 \le f(t) \le g(t)$ for all $t \in [a, b)$, and $\int_a^b f(t)dt$ is divergent, then $\int_a^b g(t)dt$ is divergent too.

b) In practice, we realize comparison with functions like in example 2.3, i.e. $\frac{1}{t^\lambda}$ on $[a, \infty)$, $\frac{1}{(b-t)^\lambda}$ on $[a, b)$, $q^t$ on $[a, \infty)$, etc. The comparison with such functions leads to particular forms of Theorem 2.12, which are very useful in practice. We mention some of them in the following theorems 2.14 - 2.18.

**2.14. Theorem – special form # I of the comparison test.** (Test based on $\lim\limits_{t\to\infty} t^\lambda f(t)$) Let $f : [a, \infty) \to \mathbb{R}^+$ be integrable on any compact from $[a, \infty)$ and let us note $\ell = \lim\limits_{t\to\infty} t^\lambda f(t)$.

1) If $\lambda > 1$ and $0 \le \ell < \infty$, then $\int_a^\infty f(t)dt$ is convergent
2) If $\lambda \le 1$ and $0 < \ell \le \infty$, then $\int_a^\infty f(t)dt$ is divergent.

Proof. If $\ell \in (0, \infty)$, then for every $\varepsilon > 0$ there exists $\delta > 0$ such that $t > \delta$ implies $0 < \ell - \varepsilon < t^\lambda f(t) < \ell + \varepsilon$, i.e.

$$\frac{\ell-\varepsilon}{t^{\lambda}} < f(t) < \frac{\ell+\varepsilon}{t^{\lambda}}.$$

If $\lambda \le 1$, then the integral of $\frac{1}{t^{\lambda}}$ on $[\delta, \infty)$ is divergent, so the first inequality from above shows that $\int_a^{\infty} f(t)dt$ is divergent too. Similarly, if $\lambda > 1$, then $\frac{1}{t^{\lambda}}$ is integrable on $[\delta, \infty)$, and the second inequality shows that the integral $\int_a^{\infty} f(t)dt$ is convergent.

The cases $\ell = 0$ and $\ell = \infty$ are similarly discussed using a single inequality from above. ◇

**2.15. <u>Theorem – special form # II of the comparison test</u>** (Test based on $\lim_{t\to b}(b-t)^{\lambda} f(t)$) Let $f : [a, b) \to \mathbb{R}^{+}$ be integrable on any compact from $[a, b)$, and let us note $\ell = \lim_{t\to b}(b-t)^{\lambda} f(t)$, where $\lambda \in \mathbb{R}$.

1) If $\lambda < 1$ and $0 \le \ell < \infty$, then $\int_a^{b} f(t)dt$ is convergent, and

2) If $\lambda \ge 1$ and $0 < \ell \le \infty$, then $\int_a^{b} f(t)dt$ is divergent.

The proof is similar to the above one, but uses the testing function $\frac{1}{(b-t)^{\lambda}}$ on $[a, b)$ . ◇

The above two tests have the inconvenient that they refer to positive functions. The following two theorems are consequences of the comparison test for the case of non-necessarily positive functions.

**2.16. <u>Theorem – special form # III of the comparison test.</u>** (Test of integrability for $f(t) = \frac{\varphi(t)}{t^{\lambda}}$ on $[a, \infty)$. Let $f : [a, \infty) \to \mathbb{R}$, where $a > 0$, be a function of the form $f(t) = \frac{\varphi(t)}{t^{\lambda}}$ where:

1) $\varphi$ is continuous on $[a, \infty)$

2) There exists $M > 0$ such that $\left|\int_a^{\alpha} \varphi(t)dt\right| \le M$ for all $\alpha > a$.

Then $\int_a^\infty f(t)dt$ is convergent, whenever $\lambda > 0$.

Proof. By hypothesis, for $\Phi(\alpha) = \int_a^\alpha \varphi(t)dt$ we have $\left|\frac{\Phi(\alpha)}{\alpha^{\lambda+1}}\right| \le \frac{M}{x^{\lambda+1}}$ for all $\alpha \in [a, \infty)$. Since $\lambda + 1 > 1$, it follows that $\int_a^\infty \frac{d\alpha}{\alpha^{\lambda+1}}$ is convergent. So, according to theorem 2.12, $\int_a^\infty \frac{\Phi(\alpha)}{\alpha^{\lambda+1}} d\alpha$ is absolutely convergent. Integrating by parts we obtain

$$\int_a^\infty \frac{\varphi(t)}{t^\lambda} dt = \int_a^\infty \Phi'(t)\frac{1}{t^\lambda} dt = \lambda\int_a^\infty \frac{\Phi(t)}{t^{\lambda+1}} dt$$

which shows that $f$ is integrable on $[a, \infty)$. ◇

**2.17. Theorem – special form # IV of the comparison test.** (Test of integrability for $f(t) = (b-t)^\lambda\varphi(t)$ on $[a, b)$). Let $f : [a, b) \to \mathbb{R}$, where $b \in \mathbb{R}$, be a function of the form $f(t) = (b-t)^\lambda\varphi(t)$. If

1) $\varphi$ is continuous on $[a, b)$
2) there exists $M > 0$ such that $\left|\int_a^\alpha \varphi(t)dt\right| \le M$ for all $\alpha \in [a, b)$,

then the integral $\int_a^b f(t)dt$ is convergent for any $\lambda > 0$.

Proof. Let us remark that $\Phi(\alpha) = \int_a^\alpha \varphi(t)dt$ verifies the inequality

$$\left|\frac{\Phi(\alpha)}{(b-\alpha)^{1-\lambda}}\right| \le \frac{M}{(b-\alpha)^{1-\lambda}}.$$

Since $1-\lambda < 1$, $\int_a^b \frac{d\alpha}{(b-\alpha)^{1-\lambda}}$ is convergent, hence $\int_a^b \frac{\Phi(\alpha)}{(b-\alpha)^{1-\lambda}}d\alpha$ is absolutely convergent. It remains to integrate by parts

$$\int_a^b (b-t)^\lambda \varphi(t)dt = \int_a^b (b-t)^\lambda \Phi'(t)dt = \lambda\int_a^b \frac{\Phi(t)}{(b-t)^{1-\lambda}} dt$$

and use the form of $f$. ◇

The following test is based on the comparison with the particular function $g : [a,\infty) \to \mathbb{R}$, of the form $g(x) = q^x$, where $q > 0$ and $a > 0$ (see also problem V.2.1).

**2.18. <u>Theorem – special form # V of the comparison test.</u>** (The Cauchy's root test) Let $f : [a, \infty) \to \mathbb{R}$, where $a > 0$, be integrable on any compact from $[a, \infty)$, and let us suppose that there exists $\ell = \lim\limits_{t\to\infty} |f(t)|^{1/t}$.

1) If $\ell < 1$, then $\int\limits_a^\infty f(t)dt$ is absolutely convergent, and

2) If $\ell > 1$, then $\int\limits_a^\infty f(t)dt$ is not absolutely convergent.

<u>Proof.</u> By the definition of $\ell$, we know that for every $\varepsilon > 0$ there exists $\delta > 0$ such that $t > \delta$ implies $|\,|f(t)|^{1/t} - \ell\,| < \varepsilon$, i.e. $\ell - \varepsilon < |f(t)|^{1/t} < \ell + \varepsilon$. If $\ell < 1$, let us note $q = \ell + \varepsilon < 1$. If $t > \delta$, we have $|f(t)| < q^t$.

So, it remains to see that $q^t$ is integrable on $[\delta, \infty)$ since $q < 1$. Because $f$ is integrable on the compact $[a, \delta]$, it will be integrable on $[a, \infty)$ too. The second case is similarly analyzed by noting $q = \ell - \varepsilon > 1$, when $\int\limits_\delta^\infty q^t dt$ is divergent, and $|f(t)| > q^t$. ◇

The convergence of some improper integrals can be reduced to the convergence of sequences and series.

**2.19. <u>Theorem</u>.** (Test of reduction to series) If $f : [a, \infty) \to \mathbb{R}^+$ is a decreasing function, integrable on any $[a, b] \subset [a, \infty)$, then the following assertions are equivalent:

a) $\int\limits_a^\infty f(t)dt$ is convergent

b) The sequence of terms $u_n = \int\limits_a^{a+n} f(t)dt$, $n \in \mathbb{N}$, is convergent

c) The series $\sum\limits_{n\in N} f(a+n)$ is convergent.

<u>Proof.</u> a) implies b) because if there exists $\ell = \lim\limits_{b\to\infty} \int\limits_a^b f(t)dt$, then $\lim\limits_{n\to\infty} \int_a^{a+n} f(t)dt = \ell$ too.

The written integrals exist because decreasing functions are integrable on compact intervals.

b) $\Rightarrow$c) follows from the inequality $f(\mathrm{t}) \geq f(a+n)$ on $[a+n-1,\ a+n]$, which leads to $\sum\limits_{k=1}^{n} f(a+k) \leq \int\limits_{a}^{a+n} f(t)dt$ .

Finally, c) $\Rightarrow$a) because from $\int\limits_{a+k-1}^{a+k} f(t)dt \leq f(a+k-1)$ it follows that $\int\limits_{a}^{b} f(t)dt \leq \sum\limits_{k=0}^{n-1} f(a+k)$ for all $b \in [a,\ a+n]$ . ◇

**2.20. <u>Remarks.</u>** a) Between improper integrals and series there are still significant differences. For example, the convergence of $\int\limits_{0}^{\infty} f(t)dt$ does not generally imply $\lim\limits_{t\to\infty} f(t) = 0$ (see problem 6) .

b) The notion of improper integral is sometimes used in a more general sense, namely that of "principle value" (also called "Cauchy's principal value"), denoted as $p.v.\int \ldots$. By definition,

$$p.v. \int\limits_{-\infty}^{+\infty} f(t)dt = \lim_{x\to\infty} \int\limits_{-x}^{x} f(t)dt \text{, and}$$

$$p.v.\int\limits_{a}^{b} f(t)dt = \lim_{\substack{\varepsilon\to0 \\ \varepsilon>0}} \left[ \int\limits_{a}^{c-\varepsilon} f(t)dt + \int\limits_{c+\varepsilon}^{b} f(t)dt \right]$$

where $c \in (a, b)$ is the point around where $f$ is unbounded.

Of course, the convergent integrals are also convergent in the sense of the principal value, but the converse implication is generally not true (see problem 7).

**PROBLEMS § V.2.**

**1.** Show that $\int_a^\infty q^t dt$, where $a > 0$, $q > 0$ is convergent for $q < 1$ and it is divergent for $q \geq 1$.

<u>Hint.</u> If $q = 1$, then $\int_a^\infty dx$ is divergent. Otherwise $\int_a^b q^x dx = \frac{1}{\ln q}[q^b - q^a]$.

**2.** Study the convergence of the integrals $\int_1^\infty \frac{\sin x}{x^3} dx$ and $\int_0^1 \ln x dx$ .

<u>Hint.</u> Use theorems 2.14 and 2.15 for $\left|\frac{\sin x}{x^3}\right|$ and $|\,ln x|$.

**3.** Show that $\int_0^\infty \frac{\sin x}{x} dx$ is convergent but not absolutely convergent.

<u>Hint.</u> Because $\lim_{x \to 0} \frac{\sin x}{x} = 1$, the integral is improper only at the upper limit. We can apply theorem 2.16 (special form # III) to $\varphi(x) = \sin x$, for $\lambda = 1$. The integral is not absolutely convergent because for $x \geq a > 0$ we have $\left|\frac{\sin x}{x}\right| \geq \frac{\sin^2 x}{x}$, and

$$\int_a^\infty \frac{\sin^2 x}{x} dx = \int_a^\infty \frac{dx}{2x} - \int_a^\infty \frac{\cos 2x}{2x} dx$$

which is divergent.

**4.** Establish the convergence of $\int_0^1 (\cos\frac{1}{x})\frac{dx}{x^{2-\lambda}}$, for $\lambda \in (0, 2)$.

<u>Hint.</u> Apply theorem 2.17 (special form # IV) for $\varphi(x) = \frac{1}{x^2}\cos\frac{1}{x}$, since

$$\left|\int_x^1 \frac{1}{t^2}\cos\frac{1}{t} dt\right| = \left|\sin\frac{1}{x} - \sin 1\right| \leq 2 .$$

**5.** Analyze the convergence of the integrals

$$I_n = \int_1^\infty \frac{x^n}{\left(\frac{1}{n}+x\right)^n}dx \text{, and } J_n = \int_1^\infty \frac{x^n}{\left(\frac{1}{n}+\frac{1}{x}\right)^n}dx,$$

where $n \in \mathbb{N}^*$.

Hint. Use theorem 2.18 (special form # V). For $I_n$, $\lim_{x\to\infty}\frac{(x^{\frac{1}{n}})^n}{\frac{1}{n}+x} = 0 < 1$, hence $I_n$ is (absolutely) convergent. For the (positive) function in $J_n$ we have $\lim_{x\to\infty}\frac{\left(x^{\frac{1}{x}}\right)^n}{\frac{1}{n}+\frac{1}{x}} = n$, so $J_n$ is divergent for $n > 1$. In the case $n = 1$, we have $\lim_{x\to\infty}\frac{x}{1+\frac{1}{x}} = \infty$, hence $J_1$ is divergent.

**6.** Show that $\int_1^\infty t\cos t^3 dt$ is convergent even if $\lim_{x\to\infty} x\cos x^3$ doesn't exist. Is this situation possible for positive functions instead of $x\cos x^3$ ?

Hint. Use theorem 2.16 for $\varphi(x) = x^2\cos x^3$ and $\lambda = 1$, since

$$\left|\int_1^x t^2\cos t^3 dt\right| = \frac{1}{3}|\sin x^3 - \sin 1| \leq \frac{2}{3}.$$

According to theorem 2.14, the answer to the question is negative, i.e. positive functions which are integrable on $[a, \infty)$ must have null limit at infinity. In fact, on the contrary case, when $\lim_{x\to\infty} f(x)$ doesn't exist or is different from zero, we have $\lim_{x\to\infty} xf(x) = \infty$, hence taking $\lambda = 1$ and $\ell = \infty$ in the mentioned test, it would follow that $\int_a^\infty f(t)dt$ is divergent.

**7.** Study the principal values of the integrals

$$I = \int_{-\infty}^\infty e^{-|t|}\sin t dt,\ J = \int_{-\infty}^\infty \left[t^{-\frac{1}{2}}\right]dt,$$

where $[x]$ is the entire part of $x$,

$$K = \int_{-\infty}^{+\infty} \cos t dt, \text{ and } L = \int_{-1}^{2} \frac{dt}{t}.$$

Solution. $I$ is (absolutely) convergent; $J$ is divergent, but $p.v.J = 0$; $K$ is divergent in the sense of $p.v.$; $L$ is divergent, but $p.v.L = ln2$.

**8.** Study the convergence of the integrals $I_n = \int_0^\infty x^n e^{-x} dx$, $J_n = \int_0^\infty \sin x^n dx$, and $K_n = \int_0^\infty \cos x^n dx$, where $n \in \mathbb{N}$.

Hint. $\lim\limits_{x\to\infty} x^{n+2} e^{-x} = 0$ for any $n \in \mathbb{N}$, hence applying theorem 2.14, $I_n$ is convergent. $J_0$, $J_1$, $K_0$, $K_1$ are divergent according to the definition. In $J_n$ and $K_n$, for $n \geq 2$ we may replace $x = \sqrt[n]{t}$, and use theorem 2.16.

**9.** Show that the following integrals have the specified values:

a) $I_n = \int_0^\infty e^{-x} \cdot x^n \, dx = n!$

b) $J_n = \int_0^\infty e^{-x^2} \cdot x^{2n+1} dx = \frac{n!}{2}$.

Hint. a) Establish the recurrence formula $I_n = n\, I_{n-1}$.
b) Replace $x^2 = t$ in the previous integral.

**10.** Using adequate improper integrals, study the convergence of the series:

a) $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}, \alpha \in \mathbb{R}_+^*$; b) $\sum_{n=1}^{\infty} \frac{\ln n}{n^\alpha}, \alpha \in \mathbb{R}$; c) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^\alpha}, \alpha \in \mathbb{R}$.

Hint. Use theorem 2.19. In $\int_1^b \frac{\ln x}{x^\alpha} dx$ we can integrate by parts. In the integral $\int_2^\infty \frac{dx}{x(\ln x)^\alpha}$ we can change $ln\ x = t$. All these integrals (and the corresponding series) are convergent iff $\alpha > 1$.

## § V.3. IMPROPER INTEGRALS WITH PARAMETERS.

We will reconsider the topic of § V.1 in the case of improper integrals.

**3.1. Definition.** Let $A \subseteq \mathbb{R}$ , $I$ = [a, b) $\subset \mathbb{R}$, and $f : A \times I \to \mathbb{R}$ be such that for each $x \in A$, the function $t \mapsto f(x, t)$ is improperly integrable on [a, b). Then $F : A \to \mathbb{R}$, expressed by

$$F(x) = \int_a^b f(x,t)dt;\ \int_a^{\infty} f(x,t)dt;\ \int_{-\infty}^{+\infty} f(x,t)dt;\ \text{etc.}$$

is called *improper integral with parameter*.

**3.2. Remark.** According to the definition of an improper integral, $F$ is defined as a point-wise limit of some definite integrals, i.e.

$$F(x) \overset{p}{=} \lim_{\beta \to b} \int_a^{\beta} f(x,t)dt\,.$$

More exactly, this means that for any $x \in A$ and $\varepsilon > 0$, there exists $\delta(x, \varepsilon) > 0$ such that for all $\beta \in (b - \delta, b)$, we have $\left| \int_a^{\beta} f(x,t)dt - F(x) \right| < \varepsilon$ .

Many times we need a stronger convergence, like the uniform one, which means that for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for all $x \in A$ and $\beta \in (b - \delta, b)$, we have the same inequality: $\left| \int_a^{\beta} f(x,t)dt - F(x) \right| < \varepsilon$ .

In this case we say that the improper integral *uniformly converges* to $F$, and we note $F(x) \overset{u}{=} \lim_{\beta \to b} \int_a^{\beta} f(x,t)dt$.

The following lemma reduces the convergence of the integral to the convergence of some function sequences and series.

**3.3. Lemma.** Let us consider $A \subseteq \mathbb{R}$ , $I$ = [a, b) $\subset \mathbb{R}$, and $f : A \times I \to \mathbb{R}$ a function, such that for each $x \in A$, the map $t \mapsto f(x, t)$ is integrable on each compact from $I$. The following assertions are equivalent:

(i) The improper integral $\int_a^b f(x,t)dt$, with parameter $x$, is uniformly (point-wise) convergent on $A$ to $F$ ;

(ii) For arbitrary increasing sequence $(\beta_n)_{n \in \mathbb{N}}$ for which $\beta_0 = a$ and $\lim_{n \to \infty} \beta_n = b$, the <u>function sequence</u> $(F_n)_{n \in \mathbb{N}}$, where $F_n : A \to \mathbb{R}$ have the values $F_n(x) = \int_a^{\beta_n} f(x,t)dt$, is uniformly (point-wise) convergent on $A$ to $F$.

(iii) For arbitrary increasing sequence $(\beta_n)_{n\in\mathbb{N}}$ such that $\beta_0 = a$ and $\lim\limits_{n\to\infty} \beta_n = b$, the function series $\sum\limits_{n=0}^{\infty} u_n$, of terms $u_n : A \to \mathbb{R}$, where

$$u_n(x) = \int_{\beta_n}^{\beta_{n+1}} f(x,t)dt,$$

is uniformly (point-wise) convergent on $A$ to $F$.

The proof is routine and will be omitted, but we recommend to follow the scheme: (i)$\Leftrightarrow$(ii) $\Leftrightarrow$(iii) .

**3.4. Theorem.** (Cauchy's general test) Let $A \subseteq \mathbb{R}$, $I = [a, b) \subset \mathbb{R}$, and $f : A \times I \to \mathbb{R}$ be such that the map $t \mapsto f(x, t)$ is integrable on each compact from $I$, for arbitrary $x \in A$. Then the improper integral $\int_a^b f(x,t)dt$ with parameter $x$, is *uniformly convergent* on $A$ iff for every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for arbitrary $x \in A$ and $b', b'' \in (b - \delta, b)$, we have

$$\left| \int_{b'}^{b''} f(x,t)dt \right| < \varepsilon .$$

Proof. If $F(x) \overset{u}{=} \lim\limits_{\beta\to b} \int_a^{\beta} f(x,t)dt$, then we evaluate

$$\left| \int_{b'}^{b''} f(x,t)dt \right| \le \left| \int_{a}^{b'} f(x,t)dt - F(x) \right| + \left| \int_{a}^{b''} f(x,t)dt - F(x) \right|$$

as we usually prove a Cauchy condition.

Conversely, using the above lemma, we show that the sequence $(F_n)_{n\in\mathbb{N}}$, where $F_n(x) = \int_a^{\beta_n} f(x,t)dt$, $\beta_0 = a$, $\beta_n < \beta_{n+1}$, and $\lim\limits_{n\to\infty} \beta_n = b$, is uniformly Cauchy on $A$. In fact, for any $\varepsilon > 0$ we have

$$|F_n(x) - F_m(x)| = \left| \int_{\beta_n}^{\beta_m} f(x,t)dt \right| < \varepsilon,$$

whenever $\beta_n, \beta_m \in (b - \delta, b)$, i.e. $m, n > n_0(\delta) \in \mathbb{N}$ . ◇

Using this general test we obtain more practical tests:

**3.5. Theorem.** (Comparison test) Let $A$, $I$ and $f$ be like in the above theorem. Let also $g : I \to \mathbb{R}^+$ be such that:

1) $|f(x, t)| \le g(t)$ for all $(x, t) \in A \times I$

2) $\int_a^b g(t)dt$ is convergent.

Then $\int_a^b f(x,t)dt$ is *uniformly convergent* on $A$.

Proof. In order to apply the above general test of uniform convergence we evaluate $\left|\int_{b'}^{b''} f(x,t)dt\right| \le \int_{b'}^{b''} |f(x,t)|dt \le \int_{b'}^{b''} g(t)dt$. The last integral can be made arbitrarily small for $b', b''$ in an appropriate neighborhood of $b$, since $g$ is integrable on [a, b). ◇

**3.6. Remark.** If compared to theorem 12, §2, we see that the uniform boundedness relative to $x$, $|f(x, t)| \le g(t)$, leads to the uniform convergence on $A$. Consequently, particular tests similar to theorems 14–18 in § V.2 are valid, if the hypothesis are uniformly satisfied relative to $x \in A$.

As in § V.1, we are interested in establishing the rules of operating with parameters in improper integrals.

**3.7. Theorem.** (Continuity of $F$) Let $f : A \times I \to \mathbb{R}$ be *continuous* on $A \times I$, where $A \subseteq \mathbb{R}$, and $I$ = [a, b) $\subset \mathbb{R}$. If the integral $\int_a^b f(x,t)dt$ is *uniformly convergent* on $A$, then $F : A \to \mathbb{R}$, expressed by $F(x) = \int_a^b f(x,t)dt$ is continuous on $A$ .

Proof. According to Lemma 3.3, $F \overset{u}{=} \lim_{n\to\infty} F_n$. On the other hand, $F_n$ are continuous on $A$ (see theorem 3 in §1). Consequently, $F$ is continuous as a uniform limit of continuous functions. ◇

**3.8. Theorem.** (Derivability of $F$) Let $A \subseteq \mathbb{R}$, $I = [a, b) \subset \mathbb{R}$, and $f : A \times I \to \mathbb{R}$ be such that:

1) $f$ is continuous on $A \times I$

2) $\dfrac{\partial f}{\partial x}$ is continuous on $A \times I$

3) $\int_a^b f(x,t)dt$ is point-wise convergent on $A$ to $F : A \to \mathbb{R}$

4) $\int_a^b \dfrac{\partial f}{\partial x}(x,t)dt$ is uniformly convergent on $A$.

Then $F$ is derivable on $A$, its derivative is $F'(x) = \int_a^b \frac{\partial f}{\partial x}(x,t)dt$, and $F'$ is continuous on $A$.

Proof. Let us note $F_n(x) = \int_a^{b_n} f(x,t)dt$, where $(b_n)_{n\in\mathbb{N}}$ is an increasing sequence for which $b_0 = a$ and $\lim_{n\to\infty} b_n = b$. According to the previous lemma 3.3, $F = \lim_{n\to\infty} F_n$ point-wise. On the other side $F_n$ is derivable as a definite integral with parameter (see theorem 5, §1), and

$$F_n'(x) = \int_a^{b_n} \frac{\partial f}{\partial x}(x,t)dt.$$

Now, using the same lemma for uniformly convergent integrals, we obtain all the claimed properties of $F$ . ◇

The operation of integration may be realized either in the proper sense (as in definite integrals), or in the improper sense.

**3.9. Theorem.** (The definite integral relative to the parameter) Let us consider $A = [\alpha, \beta] \subset \mathbb{R}$, $I = [a, b) \subset \mathbb{R}$, and $f : A \times I \to \mathbb{R}$ be such that:

1) $f$ is continuous on $A \times I$

2) $\int_a^b f(x,t)dt$ is *uniformly convergent on* $A = [\alpha, \beta]$ to $F$.

Then $F$ is integrable on $[\alpha, \beta]$ and $\int_\alpha^\beta F(x)dx = \int_a^b \left[\int_\alpha^\beta f(x,t)dx\right]dt$ .

Proof. Let $(b_n)_{n\in\mathbb{N}}$ be an increasing sequence such that $b_0 = a$ and $\lim_{n\to\infty} b_n = b$. According to Lemma 3.3, $F \overset{u}{=} \lim_{n\to\infty} F_n$, where $F_n : [\alpha, \beta] \to \mathbb{R}$ are expressed by $F_n(x) = \int_a^{b_n} f(x,t)dt$. On the other hand, according to theorem 3.3, § V.1, $F_n$ are continuous functions, hence $F$ is continuous too. So, we deduce that $F$ is integrable on $[\alpha, \beta]$, and $\int_\alpha^\beta F(x)dx = \lim_{n\to\infty} \int_\alpha^\beta F_n(x)dx$.

Now it remains to use theorem 1.7, § V.1, in order to calculate

$$\int_\alpha^\beta F_n(x)dx = \int_a^{b_n} \left[\int_\alpha^\beta f(x,t)dx\right]dt,$$

and to apply lemma 3.3 again. ◇

**3.10. <u>Theorem.</u>** (The improper integral relative to the parameter) Let us consider $A = [\alpha, \beta) \subset \mathbb{R}$, $I = [a, b) \subset \mathbb{R}$, and $f : A \times I \to \mathbb{R}$ be such that:

1) $f$ is *positive* and *continuous* on $A \times I$
2) $\int_a^b f(x,t)dt$ is *uniformly* convergent to $F$: $A \to \mathbb{R}$ on any compact from $A$
3) $\int_\alpha^\beta f(x,t)dx$ is *uniformly* convergent to $G : I \to \mathbb{R}$ on $I$
4) $\int_a^b G(t)dt$ is *convergent* .

Then $F$ is improperly integrable on $[\alpha, \beta)$, and $\int_\alpha^\beta F(x)dx = \int_a^b G(t)dt$.

<u>Proof.</u> According to the previous theorem, for each $\eta \in [\alpha, \beta)$, the function $F$ is integrable on $[\alpha, \eta]$, and $\int_\alpha^\eta F(x)dx = \int_a^b \left[ \int_\alpha^\eta f(x,t)dx \right] dt$.

Let us note by $\varphi : [\alpha, \beta] \times [a, b) \to \mathbb{R}$ the function of values

$$\varphi(\eta, t) = \begin{cases} \int_\alpha^\eta f(x,t)dx & \text{if } t \in [\alpha, \beta) \\ G(t) & \text{if } t = \beta \end{cases}$$

The third hypothesis of the theorem shows that $\varphi$ is continuous on the set $[\alpha, \beta] \times [a, b)$. On the other hand, if we note by $\Phi$: $[\alpha, \beta] \to \mathbb{R}$ the function $\Phi(\eta) = \int_a^b \varphi(\eta,t)dt$, we obtain $\Phi(\eta) = \int_\alpha^\eta F(x)dx$ for all $\eta \in [\alpha, \beta)$. Now, the problem reduces to extending this relation for $\eta = \beta$. In fact, because $f$ is positive, for all $\eta \in [\alpha, \beta)$ and $t \in [a, b)$ we have $\int_\alpha^\eta f(x,t)dx \le \int_\alpha^\beta f(x,t)dx$, i.e. $\varphi(\eta, t) \le G(t)$ . Since $\int_a^b G(t)dt$ is convergent, the comparison test shows that $\int_a^b \varphi(\eta,t)dt$ is uniformly convergent to $\Phi$. Adding the fact that $\varphi$ is continuous, theorem 3.7 shows that $\Phi$ is continuous on $[\alpha, \beta]$, hence there exists $\lim\limits_{\eta\to\beta} \Phi(\eta) = \Phi(\beta)$, i.e. $\Phi(\beta) = \int\limits_{\alpha}^{\beta} F(x)dx$. Replacing $\Phi$ and $\varphi$ by their values, we obtain the claimed formula. ◇

**3.11. Remarks.** a) Theorems 3.9 and 3.10 establish the conditions when we can change the order of integration, i.e.

$$\int\limits_{\alpha}^{\beta}\left[\int\limits_{a}^{b} f(x,t)dt\right]dx = \int\limits_{a}^{b}\left[\int\limits_{\alpha}^{\beta} f(x,t)dx\right]dt.$$

b) The condition $f$ to be positive in theorem 10 is essential. For example, if $f : [1,\infty) \times [1,\infty) \to \mathbb{R}$ is expressed by $f(x, t) = \dfrac{x-t}{(x+t)^3}$, then $| f(x, t) | \leq \dfrac{1}{x^2}$ as well as $| f(x, t) | \leq \dfrac{1}{t^2}$ for all $(x, t) \in [1,\infty) \times [1,\infty)$, hence $f$ is integrable on $[1,\infty)$ relative to $t$, and also relative to $x$. By direct calculation we find $F(x) = -\dfrac{1}{(1+x)^2}$ and $G(t) = \dfrac{1}{(1+t)^2}$. Consequently, $F$ and $G$ are also integrable on $[1,\infty)$, but

$$\int\limits_{1}^{\infty} G(t)dt = \frac{1}{2} \neq -\frac{1}{2} = \int\limits_{1}^{\infty} F(x)dx.$$

Excepting the condition of being positive, $f$ satisfies all conditions of theorem 3.10.

The integrals with parameter are useful in defining new functions. The Euler's $\Gamma$ and B functions are typical examples in this sense:

**3.12. Definition.** The function $\Gamma : (0, \infty) \to (0, \infty)$ expressed by

$$\Gamma(x) = \int\limits_{0}^{\infty} t^{x-1}e^{-t}dt$$

is called *Euler's gamma function.*

The function B: $(0, \infty) \times (0, \infty) \to (0, \infty)$ of values

$$\mathrm{B}(x, y) = \int\limits_{0}^{1} t^{x-1}(1-t)^{y-1}dt$$

is called *Euler's beta function.*

This definition makes sense because:

**3.13. Proposition.** The integrals of $\Gamma$ and B are convergent.

Proof. The integral which defines $\Gamma$ is improper both at 0 and $\infty$. Because $t^{x-1}e^{-t} \leq t^{x-1}$ for $t \in [0, 1]$, and $t^{x-1}$ is integrable if $x > 0$, it follows that the integral of $\Gamma$ is convergent at 0. This integral is convergent at $\infty$ because $t^n e^{-t}$ is integrable on $[1,\infty)$ for all $n \in \mathbb{N}$.

The integral which defines B is also improper at 0 and at 1, and, in addition, it depends on two parameters. The convergence of this integral follows from the inequality $t^{x-1}(1-t)^{y-1} \le 2[t^{x-1} + (1-t)^{y-1}]$, which holds for $t \in [0, 1]$, $x > 0$ and $y > 0$ (see the comparison test). This inequality may be verified by considering two situations:

a) If $t \in [1/2, 1)$, and $x > 0$, then $t^{x-1} \le 2$, so that in this case $t^{x-1}(1-t)^{y-1} \le 2(1-t)^{y-1} \le 2[t^{x-1} + (1-t)^{y-1}]$;

b) If $t \in (0, 1/2]$, then $(1-t) \in [1/2, 1)$, and since $y > 0$ too, we have $(1-t)^{y-1} \le 2$, and a similar evaluation holds. ◇

**3.14. <u>Theorem.</u>** Function $\Gamma$ has the following properties:

(i) it is a convex and indefinitely derivable function;

(ii) $\Gamma(x+1) = x\,\Gamma(x)$ at any $x > 0$ ;

(iii) $\Gamma(n+1) = n!$ for every $n \in \mathbb{N}$, i.e. $\Gamma$ generalizes the factorial.

<u>Proof.</u> (i) It is easy to see that $f(x, t) = t^{x-1}e^{-t}$ satisfies the conditions in theorem 3.8, hence

$$\Gamma'(x) = \int_0^\infty t^{x-1}e^{-t}\ln t\,dt .$$

By repeating this argument we obtain

$$\Gamma^{(k)}(x) = \int_0^\infty t^{x-1}e^{-t}\ln^k t\,dt$$

for any $k \in \mathbb{N}^*$, i.e. $\Gamma$ is indefinitely derivable. Its convexity follows from $\Gamma''(x) > 0$ for all $x > 0$ .

(ii) Integrating by parts we obtain we obtain

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t}dt = -\lim_{t\to\infty} t^x e^{-t} + x\int_0^\infty t^{x-1}e^{-t}dt = x\,\Gamma(x) .$$

(iii) According to (*ii*), $\Gamma(n+1) = n\,\Gamma(n) = n(n-1)\ldots 1\,\Gamma(1)$, and

$$\Gamma(1) = \int_0^\infty e^{-t}dt = 1.$$

**3.15. <u>Theorem.</u>** Function B has the properties:

(i) $B(x, y) = B(y, x)$, i.e. B is symmetric;

(ii) For any $(x, y) \in (0, \infty) \times (0, \infty)$ we have $B(x, y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$;

(iii) It has continuous partial derivatives of any order.

<u>Proof.</u> (i) Changing $t = 1-\theta$, $B(x, y)$ becomes $B(y, x)$.

(ii) Replacing $t = \dfrac{v}{1+v}$ in B, we obtain $B(x, y) = \displaystyle\int_0^\infty \frac{v^{x-1}}{(1+v)^{x+y}}dv$. On the other hand, changing $t = (1+v)u$ in $\Gamma$, it follows that

$$\Gamma(x) = (1 + v)^x \int_0^\infty u^{x-1} e^{-u(1+v)} du\,.$$

Writing this relation at $x + y$ instead of $x$, we have

$$\Gamma(x+y) \frac{1}{(1+v)^{x+y}} = \int_0^\infty u^{x+y-1} e^{-u(1+v)} du\,.$$

Amplifying by $v^{x-1}$ and integrating like in B, we obtain

$$\Gamma(x+y)\mathrm{B}(x, y) = \int_0^\infty \left[ v^{x-1} \int_0^\infty u^{x+y-1} e^{-u(1+v)} du \right] dv.$$

Using theorem 10 we change the order of the integrals and we obtain

$$\Gamma(x + y)\mathrm{B}(x, y) = \int_0^\infty \left[ u^{x+y-1} e^{-u} \int_0^\infty v^{x-1} e^{-uv} dv \right] du =$$

$$= \int_0^\infty \left[ u^{x+y-1} e^{-u} u^{-x} \Gamma(x) \right] dx =$$

$$= \Gamma(x) \int_0^\infty u^{y-1} e^{-u} du = \Gamma(x)\, \Gamma(y).$$

(iii) This property results form the similar property of Γ, taking into account the above relation between Γ and B. ◇

**3.16. <u>Remarkable integrals.</u>** a) $\Gamma(\frac{1}{2}) = \int_0^\infty \frac{e^{-t}}{\sqrt{t}} dt = \sqrt{\pi}$ and $\int_0^\infty e^{-u^2} du = \frac{\sqrt{\pi}}{2}$

(also called Euler-Poisson integral).

In fact, $\mathrm{B}(\frac{1}{2}, \frac{1}{2}) = \Gamma^2(\frac{1}{2}) = \int_0^1 \frac{dx}{\sqrt{x(1-x)}}$, which turns out to be $\pi$, if replacing $x = sin^2 t$ .

The second integral follows from $\Gamma(\frac{1}{2})$ by taking $t = u^2$ .

b) The binomial integral $I = \int_0^\infty \frac{x^m}{(a+bx^n)^p} dx$, $a > 0$, $b > 0$, $np > m + 1 > 0$ may be expressed by elementary functions only if

1) $p$ is integer
2) $\frac{m+1}{n}$ is integer (positive)
3) $p - \frac{m+1}{n}$ is integer (positive).

In fact, noting $\frac{b}{a}x^n = u$ and $k = \frac{a^{-p}}{n}\left(\frac{a}{b}\right)^{\frac{m+1}{n}}$, we obtain

$$I = k\int_0^\infty u^{\frac{m+1}{n}-1}(1+u)^{-p}\,du\,.$$

Another change of variables, namely $\frac{u}{1+u} = v$, leads to

$$I = k\int_0^1 v^{\frac{m+1}{n}-1}(1-v)^{p-\frac{m+1}{n}-1}\,dv = k\,\mathrm{B}\left(\frac{m+1}{n}, p-\frac{m+1}{n}\right) =$$

$$= k\,\frac{\Gamma\left(\frac{m+1}{n}\right)\Gamma\left(p-\frac{m+1}{n}\right)}{\Gamma(p)}\,.$$

This formula shows that in general, $I$ is expressed by $\Gamma$; in the mentioned cases $\Gamma$ reduces to factorials, so $I$ contains only elementary functions.

We recall that in the case when $\frac{m+1}{n}$ is an integer, we make the substitution $a + bx^n = t^s$, where $s$ is the denominator of the fraction $p$. Similarly, if $\frac{m+1}{n}-p$ is an integer, the evaluation of the integral may be made by the substitution $ax^{-n} + b = t^s$.

**PROBLEMS § V.3.**

**1.** Show that $F(x) = \int_0^{\infty} e^{-t}\frac{\sin xt}{t}dt$ is convergent for $x \in [0, \infty)$ and $F(x)=arctg\, x$.

Hint. The integral is improper at $\infty$; the convergence is a consequence of the comparison test, if $g(t)=\frac{\sin xt}{t}$, $t\geq 1$ (see also theorem 2.16, §V.2). By theorem 3.8, $F'(x) = \frac{1}{1+x^2}$, hence $F(x) = arctg\, x + C$. Take $x = 0$.

**2.** Calculate $I(r) = \int_0^{\pi}(1-2r\cos x + r^2)dx$, where $|r| < 1$.

Hint. The substitution $t = tg\,\frac{x}{2}$ in $I'(r)$ gives

$$I'(r) = 2\int_0^{\pi}\frac{r-\cos x}{1-2r\cos x + r^2}dx = \frac{4}{1+r}\int_0^{\infty}\frac{t^2-a}{(t^2+a^2)(1+t^2)}dt$$

where $a = \frac{1-r}{1+r} > 0$. Breaking up $\frac{1}{(t^2+a^2)(t^2+1)} = \frac{A}{t^2+1}+\frac{B}{t^2+a^2}$, where A = –B = $\frac{1}{a^2-1}$, we obtain

$$I'(r) = \frac{4}{1+r}\left[\frac{\pi}{2}-(a^2+a)\int_0^{\infty}\frac{dt}{\left(t^2+a^2\right)\left(t^2+1\right)}\right] = 0.$$

Consequently, $I(r) = C$, but $I(0) = 0$, hence $I(r) = 0$ too.

**3.** Show that $\Phi(x) = \int_0^{\infty} e^{-xt}\frac{\sin t}{t}dt = \frac{\pi}{2} - arctg x$, and deduce that $\int_0^{\infty}\frac{\sin t}{t}dt = \frac{\pi}{2}$ (Poisson).

Hint. Using the result of problem 1, $\Phi(x) = F(\frac{1}{x})$ = arctg $\frac{1}{x} = \frac{\pi}{2} - arctg\, x$.

Another method consists in integrating two times by parts in $\Phi'(x)$, and obtaining $\Phi'(x) = -1 - x^2\,\Phi'(x)$, wherefrom it follows that $\Phi(x) = -arctg\, x + C$.

For $x \to \infty$ we deduce $C = \dfrac{\pi}{2}$. Finally, the Poisson's integral is Φ(0).

**4.** Calculate $I = \displaystyle\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx$, and $J = \displaystyle\int_0^\infty \frac{\cos ax - \cos bx}{x^2} dx$, where $0 < a < b$.

Hint. $I = \displaystyle\int_0^\infty \left[\int_a^b e^{-tx} dt\right] dx = \int_a^b \left[\int_0^\infty e^{-tx} dx\right] dt = \int_a^b \frac{1}{t} dt = \ln\frac{b}{a}$.

$J = \displaystyle\int_0^\infty \left[\frac{1}{x}\int_a^b \sin txdt\right] dx = \int_a^b \left[\int_0^\infty \frac{\sin tx}{x} dx\right] dt = \frac{\pi}{2}(b - a)$, where $\displaystyle\int_0^\infty \frac{\sin tx}{x} dx = \frac{\pi}{2}$ is the Poisson's integral (see problem 3.3) independently of $t > 0$.

**5.** Let $f : (0, 1] \times (0, 1] \to \mathbb{R}$ be a function of values $f(x,t) = \dfrac{x-t}{(x+t)^3}$.

Show that $\displaystyle\int_0^1 \left[\int_0^1 \frac{y-x}{(x+y)^3} dx\right] dy = \frac{1}{2}$, $\displaystyle\int_0^1 \left[\int_0^1 \frac{y-x}{(x+y)^3} dy\right] dx = -\frac{1}{2}$, and explain why these integrals have different values.

Hint. Theorem 3.10 does not work since $f$ changes its sign.

**6.** Use the functions beta and gamma to evaluate the integrals

a) $I = \displaystyle\int_0^1 x^{p-1}(1 - x^m)^{q-1} dx$, $p, q, m > 0$;

b) $J = \displaystyle\int_0^\infty x^p e^{-x^q} dx$, $p > -1$, $q > 0$.

Hint. a) Change the variable $x^m = t$, and evaluate

$$I = \frac{1}{m}\int_0^1 t^{\frac{p}{m}-1}(1-t)^{q-1} dt = \frac{1}{m} \cdot B\left(\frac{p}{m}, q\right).$$

b) Replace $x^q = t$, and calculate

$$J = \frac{1}{q}\int_0^\infty t^{\frac{p+1}{q}-1} \cdot e^{-t} dt = \frac{1}{q} \cdot \Gamma\left(\frac{p+1}{q}\right).$$

# CHAPTER VI. LINE INTEGRAL

We will generalize the usual definite integral in the sense that instead of functions defined on $[a, b) \subset \mathbb{R}$ we will consider functions defined on a segment of some curve. There are two kinds of line integrals, depending of the considered function, which can be a scalar or vector function, but first of all we must precise the terminology concerning curves (there are plenty materials in the literature).

## § VI.1. CURVES

We analyze the notion of curve in $\mathbb{R}^3$, but all the notions and properties can be obviously transposed in $\mathbb{R}^p$, $p \in \mathbb{N} \setminus \{0, 1\}$, in particular in $\mathbb{R}^2$.

**1.1. <u>Definition.</u>** The set $\gamma \subset \mathbb{R}^3$ is called *curve* iff there exists $[a, b] \subset \mathbb{R}$ and a function $\varphi : [a, b] \to \mathbb{R}^3$ such that $\gamma = \varphi([a, b])$. In this case $\varphi$ is called *parameterization* of $\gamma$.

**1.2. <u>Types of curves.</u>** The points $A = \varphi(a)$ and $B = \varphi(b)$ are called *end-points* of the curve $\gamma$ ; if $A = B$, we say that $\gamma$ is *closed*.

We say that $\gamma$ is *simple* (without loops) iff $\varphi$ is injective.

Curve $\gamma$ is said to be *rectifiable* iff $\varphi$ has *bounded variation*, i.e. there exists

$$\overset{b}{\underset{a}{V}} \varphi = \sup_{\delta} \left( \sum_{i=0}^{n-1} \left\| \varphi(t_{i+1}) - \varphi(t_i) \right\| \right),$$

where $\delta = \{t_0 = a < t_1 < \ldots < t_n = b\}$ is a *division* of $[a, b]$. The number $L = \overset{b}{\underset{a}{V}} \varphi$ is called *length* of $\gamma$.

We say $\gamma$ is *continuous* (*Lipschitzean*, etc.) iff $\varphi$ is so.

Let us note $\varphi(t) = (x(t), y(t), z(t))$ for any $t \in [a, b]$. If $\varphi$ is differentiable on $[a, b]$, and $\varphi'$ is continuous and non-null, we say that $\gamma$ is a *smooth* curve. This means that there exist continuous derivatives $x'$, $y'$ and $z'$, and

$$x'^2(t) + y'^2(t) + z'^2(t) \neq 0 \,, \ \forall t \in [a, b] \,.$$

The vector $\vec{t}\,(x'(t), y'(t), z'(t))$ is called *tangent* to $\gamma$, at $M_0(x(t_0), y(t_0), z(t_0))$.

For practical purposes, we frequently deal with *continuous* and *piece-wise smooth* curves, i.e. curves for which there exists a finite number of intermediate points $C_k \in \gamma$, $k = \overline{1, n}$, where $C_k = \varphi(c_k)$ for some $c_k \in (a, b)$, such that $\varphi$ is smooth on each of $[a, c_1]$, on $[c_k, c_{k+1}]$ for all $k = 1, \ldots, n-1$, and on $[c_n, b]$, and $\varphi$ is continuous on $[a, b]$. The image of a restriction of $\varphi$ to $[c, d] \subseteq [a, b]$ is called *sub-arc* of the curve $\gamma$, so $\gamma$ is piece-wise smooth iff it consists of a finite number of smooth sub-arcs.

**1.3. <u>Remarks</u>.** The class of rectifiable curves is very important since it involves the notion of *length*. Geometrically speaking, the sum

$$\sum_{i=0}^{n-1} \|\varphi(t_{i+1}) - \varphi(t_i)\| ,$$

from the above definition of the variation $\overset{b}{\underset{a}{V}}\varphi$, represents the length of a broken line of vertices $\varphi(t_i)$. Passing to *finer* divisions of $\gamma$ leads to longer broken lines, hence $\gamma$ is rectifiable iff the family of these inscribed broken lines has un upper bound for the corresponding lengths.

Without going into details, we mention that a function $f:[a,b]\to\mathbb{R}$ has bounded variation if it has one of the following properties: monotony, Lipschitz property, bounded derivative, or it is a primitive, i.e. $f(x)=\int_a^x \omega(t)dt$, $\forall x\in[a,b]$ (for details, including properties of the functions with bounded variation, see [FG], [N-D-M], etc.). The above definition of the rectifiable curves is based on the following relation between *bounded variation* and *length* of a curve:

**1.4. <u>Theorem</u> (Jordan).** Let $\varphi = (\alpha, \beta)$: $[a, b]\to\mathbb{R}^2$ be a parameterization of a plane curve $\gamma$. The curve $\gamma$ is rectifiable if and only if the components $\alpha$, and $\beta$ of $\varphi$ have bounded variation.

We omit the proof, but the reader may consult the same bibliography.

**1.5. <u>Corollary</u>.** If $\gamma$ is a smooth curve, then it is rectifiable, and its length is

$$L=\int_a^b \sqrt{\alpha'^2(t)+\beta'^2(t)}\ dt .$$

A similar formula holds for curves in $\mathbb{R}^3$ and $\mathbb{R}^n$.

Because all the notions from above are based on some parameterization, it is important to know how can we change this parameterization, and what happens when we change it. These problems are solved by considering the following notion of "equivalent" parameterizations of a smooth curve.

**1.6. <u>Definition.</u>** The functions $\varphi : [a, b] \to \mathbb{R}^3$ and $\psi : [c, d] \to \mathbb{R}^3$ are *equivalent* parameterizations iff there exists a diffeomorphism

$$\sigma : [a, b] \to [c, d]$$

such that $\sigma'(t) \neq 0$ for all $t \in [a, b]$, and $\varphi = \psi \circ \sigma$. In this case we usually note $\varphi \approx \psi$, and we call $\sigma$ an *intermediate* function.

**1.7. <u>Remarks.</u>** (i) Relation $\approx$ from above is really an equivalence. In addition, this equivalence is appropriate to parameterizations of a curve because equivalent functions have identical images. When we are interested in studying more general than smooth curves, the "intermediate" function $\sigma$ (in definition 1.3) satisfies less restrictive conditions, as for example, it can only be a topological homeomorphism.

(ii) Because $\sigma : [0, 1] \to [a, b]$ defined by $\sigma(t) = tb + (1-t)a$, is an example of intermediate (even increasing) function in definition 1.3, we can always consider the curves as images of [0, 1] through continuous, smooth or other functions.

Another useful parameterization is based on the fact that the function $\sigma : [a, b] \to [0, L]$, defined by $\sigma(t) = \int_a^t \sqrt{x'^2(\theta) + y'^2(\theta) + z'^2(\theta)}d\theta$ satisfies the conditions of being an intermediate function. In this case $s = \sigma(t)$ represents the length of the sub-arc corresponding to $[a, t]$, and $L$ is the length of the whole arc $\gamma$. If $s$ is the parameter on a curve, we say that the curve is given in the *canonical* form.

(iii) From a pure mathematical point of view a curve is a class of equivalent functions. In other words we must find those properties of a curve, which are invariant under the change of parameters. More exactly, a property of a curve is an *intrinsic* property iff it does not depend on parameterization in the class of equivalent functions (the sense of the considered equivalence defines the type of property: continuous, smooth, etc.). For example, the properties of a curve of being closed, simple, continuous, Lipschitzean, and smooth are intrinsic. Similarly, the length of a curve should be an intrinsic property, so that the following result is very useful:

**1.8. Proposition.** The property of a curve of being rectifiable and its length do not depend on parameterization.

Proof. Being monotonic, $\sigma$ realizes a 1:1 correspondence between the divisions of $[a, b]$ and $[c, d]$, such that the variation of the equivalent functions on corresponding divisions are equal. It remains to recall that the length is obtained as a supremum. ◇

The fact that either $\sigma' > 0$ or $\sigma' < 0$ in definition 3 allows us to distinguish two subclasses of parameterizations which define the orientation of a curve.

**1.9. Orientated curves.** To orientate a curve means to split the class of equivalent parameterizations into two subclasses, which consist of parameterizations related by *increasing* intermediate functions, and to choose which of these two classes represent the *direct* orientation (sense), and which is the *converse* one.

By convention, the direct (positive) sense on a closed, simple and smooth curve in the Euclidean plane is the *anti-clockwise* one. More generally, the closed curves on orientated surfaces in $\mathbb{R}^3$ are directly orientated if the positive normal vector leaves the interior on its left side when running in the sense of the curve.

Alternatively, instead of considering *two senses* on a curve, we can consider *two orientated* curves. More exactly, if $\gamma$ is an orientated curve (i.e. the intermediate diffeomorphism in definition 1.3 is also increasing) of parameterization $\varphi : [a, b] \to \mathbb{R}^3$, then the curve denoted $\gamma^-$, of parameterization $\psi : [a, b] \to \mathbb{R}^3$ defined by $\psi(t) = \varphi(a + b - t)$ is called the *opposite* of $\gamma$.

Another way of expressing the orientation on a curve is that of defining an *order* on it. More exactly, we say that $X_1 = \varphi(t_1)$ is "before" $X_2 = \varphi(t_2)$ on $\gamma^-$ iff $t_1 \leq t_2$ on $[a, b]$. Using this terminology, we say that $A = \varphi(a)$ is the first and $B = \varphi(b)$ is the last point of the curve. If no confusion is possible, we can note $\gamma = \overset{\frown}{AB}$ and $\gamma^- = \overset{\frown}{BA}$. Contrarily to the division of a curve into sub-arcs, we can construct a curve by linking together two (or more) curves with common end-points.

**1.10. <u>Definition</u>.** Let $\gamma_i$, $i = 1, 2$ be two curves of parameterization $\varphi_i : [a_i, b_i] \to \mathbb{R}^3$ such that $\varphi_1(b_1) = \varphi_2(a_2)$. The curve $\gamma$, of parameterization $\varphi : [a_1, b_1 + (b_2 - a_2)] \to \mathbb{R}^3$, where

$$\varphi(t) = \begin{cases} \varphi_1(t) & \text{if } t \in [a_1, b_1] \\ \varphi_2(t - b_1 + a_2) & \text{if } t \in [b_1, b_1 + (b_2 - a_2)] \end{cases}$$ is called *concatenation* (union) of $\gamma_1$ and $\gamma_2$, and it is noted by $\gamma = \gamma_1 \cup \gamma_2$.

**1.11. <u>Proposition.</u>** The concatenation is an associative operation with curves having common end-points, but it is not commutative.

The proof is routine, and will be omitted. If $\gamma_1 \cup \gamma_2$ makes sense, then the concatenation $\gamma_2^- \cup \gamma_1^-$ is possible, but generally $\gamma^-_1 \cup \gamma^-_2$ is not.

**1.12. <u>Proposition.</u>** The smooth curves have tangent vectors at each $M_0 \in \gamma$, continuously depending on $M_0$. The directions of tangent vectors do not depend on parameterizations. In canonical parameterization, each *tangent* $\vec{t} = (x'(s), y'(s), z'(s))$ is a unit vector.

<u>Proof.</u> If function $\varphi : [a, b] \to \mathbb{R}^3$, of values $\varphi(t) = (x(t), y(t), z(t))$ is a parameterization of $\gamma$, then $\overrightarrow{M_0M} = (x(t) - x(t_0), y(t) - y(t_0), z(t) - z(t_0))$. Since $\varphi$ is differentiable, $\overrightarrow{M_0M} \approx (x'(t_0)(t - t_0), y'(t_0)(t - t_0), z'(t_0)(t - t_0))$, with equality when $t \to t_0$. Consequently the direction of $\vec{t}$ is given by $(x'(t_0), y'(t_0), z'(t_0))$. By changing the parameter, $t = \sigma(\theta)$, this vector multiplies by $\sigma'(\theta_0) \neq 0$, hence it will keep up the direction. For the canonical parameterization we have $\Delta s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$, hence the length of the tangent vector is $x'^2(s) + y'^2(s) + z'^2(s) = 1$. ◇

**PROBLEMS §VI.1.**

**1.** Is the graph of a function $f : [a, b] \to \mathbb{R}$ a curve in $\mathbb{R}^2$ ? Conversely, is any curve in $\mathbb{R}^2$ a graph of such function?
Hint. Each function $f$ generates a parameterization $\varphi : [a, b] \to \mathbb{R}^2$ of the form $\varphi(t) = (t, f(t))$. The circle is a curve, but not a graph.

**2.** Show that the concatenation of two smooth curves is a continuous piecewise smooth curve, but not necessarily smooth.
Hint. Use definition 1.7 of concatenation. Interpret the graph of $x \mapsto |x|$, where $x \in [-1, +1]$, as a concatenation of two smooth curves.

**3.** Let $\gamma_i$, $i = 1,2$ be two curves of parameterization $\varphi_i : [a_i, b_i] \to \mathbb{R}^3$ with common end-points, i.e. $\varphi_1(a_1) = \varphi_2(a_2)$ and $\varphi_1(b_1) = \varphi_2(b_2)$. Show that both $\gamma^{-}_1 \cup \gamma^{-}_2$ and $\gamma^{-}_2 \cup \gamma^{-}_1$ make sense and they are contrarily oriented closed curves.

**4.** Find the tangent of a plane curve implicitly given by $F(x, y) = 0$. In particular, take the case of the circle.
Hint. If $x = x(t)$, $y = y(t)$ is a parameterization of the curve, from $F(x(t), y(t)) \equiv 0$ on $[a, b]$, we deduce $dF = 0$, hence $F'_x x' + F'_y y' = 0$. Consequently, we can take $\vec{t} = (x'(t), y'(t)) = \lambda(F'_y, -F'_x)$.

**5.** If the plane curve $\gamma$ is implicitly defined by $F(x, y) = 0$, we say that $M_0 \in \gamma$ is a *critical point* iff $F'_x(M_0) = F'_y(M_0) = 0$. Study the form of $\gamma$ in the neighborhood of a critical point according to the sign of

$$\Delta = {F''_{xy}}^2 - F''_{xx} F''_{yy}.$$

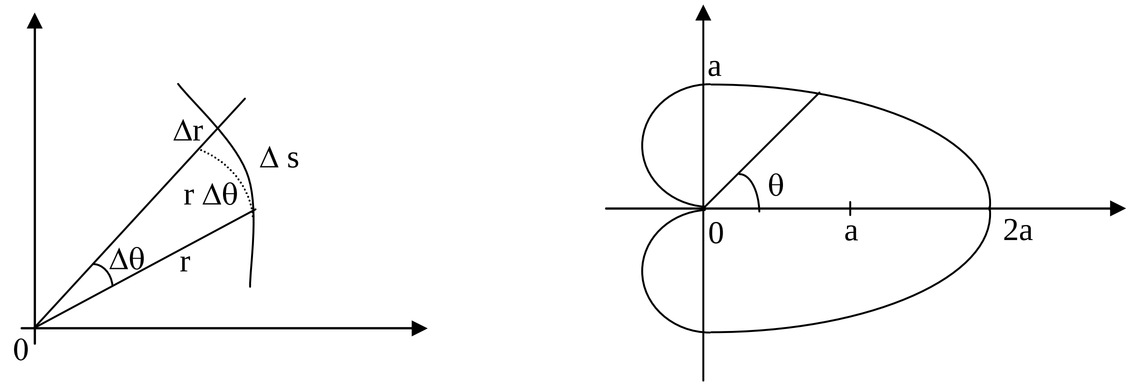
Example $y^2 = ax^2 + y^3$ , and $M_0 = (0, 0)$.
Hint. $M_0$ is a stationary point of the function $z = F(x, y)$, and $\gamma$ is the intersection of the plane $xOy$ with the surface of equation $z = F(x, y)$. In this instance $F(x_0 + h, y_0 + k) \cong F''_{xx}(x_0, y_0)h^2 + 2F''_{xy}(x_0, y_0)hk + F''_{yy}(x_0, y_0)k^2$, hence $\Delta < 0$ leads to an isolated point of $\gamma$, $\Delta > 0$ corresponds to a node (double point), and $\Delta = 0$ is undecided (isolated point). In the example, $M_0$ is isolated for $a < 0$, it is a node for $a > 0$; it is a cusp for $a = 0$ .

**6.** Find the length of the logarithmic spiral $\varphi(t) = (e^{-t}\cos t, e^{-t}\sin t, e^{-t})$, where $t \geq 0$.

Solution. $L = \int_0^\infty \sqrt{x'^2 + y'^2 + z'^2}\, dt = \sqrt{3}$.

**7.** Establish the formula of the length of a plane curve which is implicitly defined in polar coordinates, $r = r(\theta)$. Use this formula in order to find the length of the cardioid $r = a(1 + \cos\theta)$.
Hint. Following Fig. VI.1.1.a, we have

$$\Delta s^2 = (r\Delta\,\theta)^2 + (\Delta r)^2 \cong \left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]\Delta\theta^2 .$$




a) b)

Fig. VI.1.1

The length of the cardioid (sketched in Fig. VI.1.1.b) is

$$L = 2\int_0^{\pi}\sqrt{r^2 + r'^2}\,d\theta = 2a\sqrt{2}\int_0^{\pi}\sqrt{1+\cos\theta}\,d\theta = 4a\int_0^{\pi}\cos\frac{\theta}{2}\,d\theta = 8a .$$

**8.** Find the length of the curves defined by the following equations:

a) $r = a\sin^3\frac{\theta}{3}$, $\theta \in [0,2\pi]$;

b) $r = |\sin\theta|$, $\theta \in [0,2\pi]$.

Answer. a) $\frac{a}{8}(8\pi - 3\sqrt{3})$; b) $2\pi$.

**9.** Find the length of the curve of equation $\theta = \frac{1}{2}\left(r + \frac{1}{r}\right)$, $r \in [1,3]$.

Hint. Establish a formula similar to that in the above Problem 7. The length of the curve is $2 + \frac{1}{2}\ln 3$.

§ VI.2. LINE INTEGRALS OF THE FIRST TYPE

In this paragraph we consider the line integral of a scalar function. Such integrals occur in the evaluation of the mass, center of gravity, moment of inertia about an axis, etc., of a material curve with specified density.

**2.1. <u>The construction of the integral sums.</u>** Let $\gamma$ be a smooth and orientated curve in $\mathbb{R}^3$, of end-points $A$ and $B$. By a *division* of $\gamma$ we understand a set $\delta = \{M_k \in \gamma : k = 0, 1, \ldots, n\}$ such that $M_0 = A$, $M_n = B$, and $M_k < M_{k+1}$ in the order of $\gamma$, for all $k = 0, 1, \ldots, n - 1$. The *norm* of $\delta$ is $\|\delta\| = \max\limits_k \left\| \overline{M_k M_{k+1}} \right\|$.

If $\gamma_k = \overset{\frown}{M_k M_{k+1}}$ denotes the sub-arc of the end-points $M_k$ and $M_{k+1}$ on $\gamma$, we write $\Delta s_k$ for the length of $\gamma_k$, $k = 0, 1, \ldots, n - 1$. On each sub-arc $\gamma_k$ we choose a point $P_k$ between $M_k$ and $M_{k+1}$ in the order of $\gamma$. The set $\mathscr{S} = \{P_k \in \gamma_k : k = 0, 1, \ldots, n - 1\}$ represents the so called *system of intermediate points*.

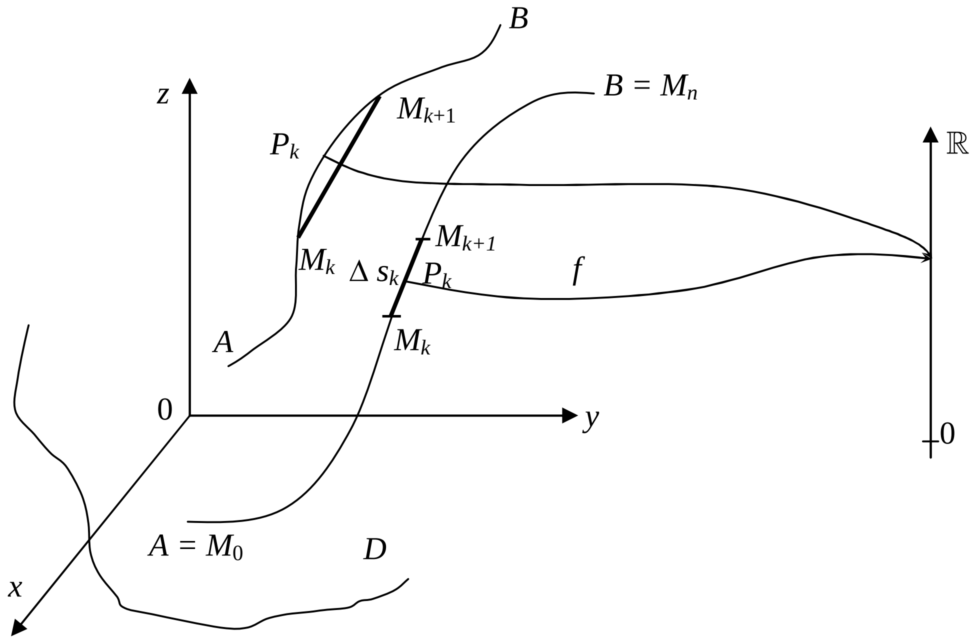



Fig. VI.2.1.

Now we consider that $\gamma$ is entirely contained in the domain $D$ on which the scalar function $f$ is defined (see Fig. VI.2.1). Under these conditions, we can calculate

$$S_{\gamma, f}(\delta, \mathscr{S}) = \sum_{k=0}^{n-1} f(P_k) \Delta s_k ,$$

which is called *integral sum* of the first type of $f$ on the curve $\gamma$, corresponding to the division $\delta$, and to the system $\mathscr{S}$ of intermediate points.

**2.2. <u>Definition.</u>** We say that $f$ is *integrable* on the curve $\gamma$ iff the above integral sums have a (finite) limit when the norm $\|\delta\| \to 0$, and this limit is not depending on the sequence of divisions with this property, and on the systems of intermediate points. If this limit exists, we note

$$\lim_{\|\delta\|\to 0} S_{\gamma,f}(\delta, \mathscr{S}) = \int_{\gamma} f ds,$$

and we call it *line integral of the first type* of $f$ on the curve $\gamma$ .

**2.3. <u>Remark.</u>** The above definition of the line integral makes no use of parameterizations, but concrete computation needs a parameterization in order to reduce the line integral to a usual Riemann integral on $\mathbb{R}$. In fact, if $\varphi : [a, b] \to \mathbb{R}^3$ is a parameterization of $\gamma$, then to each division $\delta$ of $\gamma$ there corresponds a division $d$ of $[a, b]$, defined by $M_k = \varphi(t_k)$ for all $k = 0, \ldots, n-1$. Of course, $\|d\| \to 0$ iff $\|\delta\| \to 0$. Similarly, to each system $\mathscr{S} = \{M_k \in \gamma_k : k = 0, 1, \ldots, n-1\}$ of intermediate points of $\gamma$, there corresponds a system $\mathscr{T} = \{\theta_k \in [t_k, t_{k+1}] : k = 0, 1, \ldots, n-1\}$ of intermediate points of $[a, b]$. The values $f(P_k)$ may be expressed by $(f \circ \varphi)(\theta_k)$, such that

$$S_{\gamma,f}(\delta, \mathscr{S}) = \sum_{k=0}^{n-1} (f \circ \varphi)(\theta_k)\Delta s_k =$$

$$= \sum_{k=0}^{n-1} f(x(\theta_k), y(\theta_k), z(\theta_k)) \int_{t_k}^{t_{k+1}} \sqrt{x'^2(t) + y'^2(t) + z'^2(t)}dt .$$

Finally, using the mean theorem for the above integrals, we obtain

$$S_{\gamma,f}(\delta, \mathscr{S}) = \sum_{k=0}^{n-1} (f \circ \varphi)(\theta_k)\sqrt{x'^2(\widehat{\theta}_k) + y'^2(\widehat{\theta}_k) + z'^2(\widehat{\theta}_k)}(t_{k+1} - t_k),$$

which looks like an integral sum of a simple Riemann integral. Thus we are led to the following assertion:

**2.4. <u>Theorem.</u>** Let $\gamma$ be a (simple) <u>smooth</u> curve in $D \subseteq \mathbb{R}^3$, and let $f : D \to \mathbb{R}$ be a <u>continuous</u> scalar function. Then there exists the line integral of $f$ on $\gamma$ , and for any parameterization $\varphi : [a, b] \to \mathbb{R}^3$ of $\gamma$ we have

$$\int_{\gamma} f ds = \int_a^b (f \circ \varphi)(t)\|\varphi'(t)\|dt.$$

In particular, the line integral does not depend on parameterization.

<u>Proof.</u> Let us note $F(t) = (f \circ \varphi)(t)\| \varphi'(t)\|$, and let

$$\sigma_F(d, \mathscr{T}) = \sum_{k=0}^{n-1} (f \circ \varphi)(\theta_k)\|\varphi'(\theta_k)\|(t_{k+1} - t_k)$$

be the Riemann integral sum of $F$ on $[a, b]$. Because $\gamma$ is smooth, it follows that $F$ is continuous, hence there exists $\int_a^b F(t)dt = \lim_{\|d\|\to 0} \sigma_F(d, \mathscr{T})$. More exactly, for every $\varepsilon > 0$ there exists $\eta_1 > 0$ such that for every division $d$ of $[a, b]$, for which $\|d\| < \eta_1$, we have

$$| \sigma_F(d, \mathscr{T}) - \int_a^b F(t)dt \,| < \frac{\varepsilon}{2}. \qquad (*)$$

On the other hand, $f \circ \varphi$ is uniformly continuous on the compact $[a, b]$, hence for any $\varepsilon > 0$ there exists $\eta_2 > 0$ such that for all $t', t'' \in [a, b]$ for which $| t' - t''| < \eta_2$, we have $| (f\circ \varphi)(t') - (f\circ \varphi)(t'') | < \frac{\varepsilon}{2L}$, where $L$ is the length of $\gamma$. If $d$ is a division of $[a, b]$ such that $\|d\| < \eta_2$, then

$$|S_{\gamma, f}(\delta , \mathscr{S}) - \sigma_F(d, \mathscr{T})| =$$

$$= \left| \sum_{k=0}^{n-1} \left[ (f \circ \varphi)(\theta_k) - (f \circ \varphi)(\hat{\theta}_k) \right] \cdot \left\| \varphi'(\hat{\theta}_k)(t_{k+1} - t_k) \right\| \right| \le \frac{\varepsilon}{2L} \sum_{k=0}^{n-1} \Delta s_k \le \frac{\varepsilon}{2}. \quad (**)$$

Consequently, if $d$ is a division of $[a, b]$ for which $\|d\| < \eta = \min \{\eta_1, \eta_2\}$, then using (*) and (**) we obtain

$$|S_{\gamma, f}(\delta, \mathscr{S}) - \int_a^b F(t)dt \,| \le$$

$$\le | S_{\gamma, f} (\delta, \mathscr{S}) - \sigma_F (d, \mathscr{T}) | + |\sigma_F (d, \mathscr{T}) - \int_a^b F(t)dt \,| < \varepsilon ,$$

i.e. $\int_a^b F(t)dt$ is the limit of the integral sum of $f$ on $\gamma$.

The last statement of the theorem follows from the fact that the integral sums $S_{\gamma, f}(\delta, \mathscr{S})$ do not depend on the parameterization, and the parameterization used in the construction of $F$ is arbitrary. ◇

The general properties of the line integral of the first type are summarized in the following :

**2.5. <u>Theorem.</u>** (i) The line integral of the first type is a linear functional, i.e. for any smooth curve $\gamma$, continuous $f, g$, and $\lambda, \mu \in \mathbb{R}$, we have

$$\int_\gamma (\lambda f + \mu g) ds = \lambda \int_\gamma f ds + \mu \int_\gamma g ds .$$

(ii) The line integral is additive relative to the arc, i.e.

$$\int_\gamma f ds = \int_{\gamma_1} f ds + \int_{\gamma_2} f ds, \text{ whenever } \gamma = \gamma_1 \cup \gamma_2.$$

(iii) The line integral of the first order does not depend on the orientation on the curve, i.e.

$$\int_\gamma f ds = \int_{\gamma^-} f ds .$$

The proof is directly based on definition 2.2, and will be omitted.

**PROBLEMS §VI.2.**

**1.** Calculate $\int_\gamma (x + y + z)ds$, where $\gamma$ (spiral) has the parameterization $\varphi : [0, 2\pi] \to \mathbb{R}^3$, $\varphi(t) = (\cos t, \sin t, t)$.
<u>Answer.</u> $2\sqrt{2}\ \pi^2$.

**2.** Evaluate the integral $\int_\gamma (x+y)ds$, where $\gamma$ is the curve of equation

$$(x^2 + y^2)^2 = a^2(x^2 - y^2),\ x \geq 0.$$

<u>Hint</u>. Recognize the lemniscate in polar coordinates $r = a\sqrt{\cos 2\theta}$, and use the parameterization

$$\begin{cases} x = a\sqrt{\cos 2\theta} \cdot \cos\theta \\ y = a\sqrt{\cos 2\theta} \cdot \sin\theta \end{cases},\ \theta \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right].$$

The answer is $a^2\sqrt{2}$.

**3.** Calculate the mass of the ellipse of semi-axes $a$ and $b$, which has the linear density equal to the distance of the current point up to the $x$–axis.
<u>Hint.</u> The recommended parameterization is given by $\varphi : [0, 2\pi] \to \mathbb{R}^2$, where $\varphi(t) = (a\cos t, b\sin t)$. We must calculate

$$\int_\gamma |y|ds = 2b^2 + \frac{2ab}{e}\arcsin e,$$

where $e = \frac{1}{a}\sqrt{a^2 - b^2}$ is the ex-centricity of the ellipse.

**4.** Determine the center of gravity of a half-arc of the homogeneous cycloid $x = a(t \sin t)$, $y = a(1 - \cos t)$, where $t \in [0, \pi]$.
<u>Hint.</u> $x_G = \frac{1}{M}\int_\gamma x\rho(x, s)ds$, $y_G = \frac{1}{M}\int_\gamma y\rho(x, s)ds$, where $M$ is the mass of the wire. In this case $x_G = y_G = \frac{4}{3}a$.

**5.** Find the moment of inertia, about the $z$–axis of the first loop of the homogeneous spiral $x = a \cos t$, $y = a \sin t$, $z = bt$.
<u>Hint.</u> $I_z = \int_\gamma (x^2 + y^2)\,\rho(x, y, z)\,ds = 2\,\pi a^2\sqrt{a^2 + b^2}$.

**6.** A mass $M$ is uniformly distributed along the circle $x^2 + y^2 = a^2$ in the plane $z = 0$. Find the force with which this mass acts on a mass $m$, located at the point $V(0, 0, b)$.

Hint. Generally speaking, $\vec{F} = k\dfrac{Mm}{r^3}\vec{r}$. In the particular case $\vec{F} = (0, 0, F_z)$, where $F_z = km\displaystyle\int_\gamma \frac{(z-z_0)\rho(x,y,t)}{r^3}ds = \frac{kmMb}{(a^2+b^2)^{3/2}}$.

**7.** Let $\gamma$ be an arc of the astroid in the first quadrant, whose local density equals the cube of the distance to the origin. Find the force of attraction exerted by $\gamma$ on the unit mass placed at the origin.

Hint. A parameterization of the astroid is $x = a\cos^3 t$, $y = a\sin^3 t$. Up to a constant $k$, which depends on the chosen system of units, the components of the force have the expressions:

$$F_x = k\cdot 1\int_\gamma xds = k\int_0^{\pi/2} 3a\sin t\cos^4 t\,dt = \frac{3ak}{5};$$

$$F_y = k\cdot 1\int_\gamma yds = k\int_0^{\pi/2} 3a\sin^4 t\cos t\,dt = \frac{3ak}{5}.$$

**8.** Show that if $f$ is continuous on the smooth curve $\gamma$, of length $L$, then there exists $M^* \in \gamma$ such that the mean value formula holds

$$\int_\gamma fds = \mathrm{L}f(M^*).$$

Hint. Using a parameterization of $\gamma$, we reduce the problem to the mean value formula for a Riemann integral.

**9.** Show that if $f$ is continuous on the smooth curve $\gamma$, then

$$\left|\int_\gamma fds\right| \le \int_\gamma |f|ds.$$

Hint. Use theorem 2.4.

## § VI.3. LINE INTEGRALS OF THE SECOND TYPE

The main object of this paragraph will be the line integral of a vector function along a curve in $\mathbb{R}^3$. The most significant physical quantity of this type is the work of a force.

**3.1. <u>The construction of the integral sums</u>.** Let $\gamma \subset \mathbb{R}^3$ be a <u>smooth</u> orientated curve, and let $\vec{F} : D \to \mathbb{R}^3$ be a vector function. We suppose that $\gamma \subset D$, and that $\vec{F}$ has the components $P, Q, R : D \to \mathbb{R}$, i.e. for every $(x, y, z) \in D$, we have $\vec{F}(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z))$.

Alternatively, using the canonical base $\{\vec{i}, \vec{j}, \vec{k}\}$ of $\mathbb{R}^3$ (see Fig. VI.3.1), we obtain $\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$ and $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$.

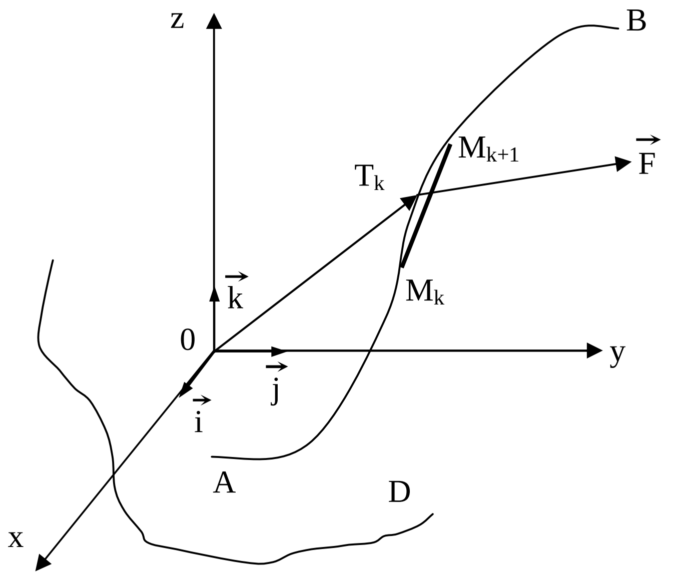



Fig. VI.3.1

If $\delta = \{M_k \in \gamma : k = 0, \ldots, n\}$ is a division of $\gamma$, we note $\vec{r}_k$ for the position vector of $M_k$. For each system of intermediate points

$$\mathscr{S} = \{T_k = (\xi_k, \eta_k, \zeta_k) \in \overset{\frown}{M_k M_{k+1}} : k = 0, \ldots, n-1\}$$

we construct the integral sum

$$S_{\gamma,\vec{F}}(\delta, \mathscr{S}) = \sum_{k=0}^{n-1} < \vec{F}(T_k), \vec{r}_{k+1} - \vec{r}_k >=$$

$$= \sum_{k=0}^{n-1} [P(\xi_k, \eta_k, \zeta_k)(x_{k+1} - x_k) + Q(\xi_k, \eta_k, \zeta_k)(y_{k+1} - y_k) + R(\xi_k, \eta_k, \zeta_k)(z_{k+1} - z_k)]$$

where $< \cdot, \cdot >$ is the Euclidean scalar product on $\mathbb{R}^3$. These sums are called *integral sums* of the *second typ*e of $\vec{F}$ along the curve $\gamma$.

**3.2. Definition.** We say that $\vec{F}$ is *integrable* on $\gamma$ iff the integral sums of the second type have a (finite) limit when the norm of $\delta$ tends to zero, and this limit is independent of the sequence of division which have $\|\delta\| \to 0$, and of the systems of intermediate points. In this case we note the limit by

$$\lim_{\|\delta\|\to 0} S_{\gamma,\vec{F}}(\delta, \mathscr{S}) = \int_\gamma <\vec{F}, d\vec{r}> = \int_\gamma \vec{F}\, d\vec{r} = \int_\gamma Pdx + Qdy + Rdz$$

and we call it *line integral of the second type* of $\vec{F}$ on $\gamma$ .

**3.3. Remark.** The main problem is to show that such integrals are also independent of the parameterization of $\gamma$, and to calculate them using parameterizations. We will solve this problem by reducing the integral of the second type to an integral of the first type, which is known how to be handled. In order to find the corresponding scalar function, we modify the form of the integral sums by using a parameterization $\varphi : [a, b] \to \mathbb{R}^3$ of $\gamma$. In fact, if $\varphi(t) = (x(t), y(t), z(t))$, then according to Lagrange's theorem, on each $[t_k, t_{k+1}]$ we have

$$x(t_{k+1}) - x(t_k) = x'(\theta_k^x)(t_{k+1} - t_k)$$
$$y(t_{k+1}) - y(t_k) = y'(\theta_k^y)(t_{k+1} - t_k)$$
$$z(t_{k+1}) - z(t_k) = z'(\theta_k^z)(t_{k+1} - t_k),$$

where $\theta_k^x, \theta_k^y, \theta_k^z \in (t_k, t_{k+1})$. Consequently, $S_{\gamma,\vec{F}}(\delta, \mathscr{S})$ becomes

$$\sum_{k=0}^{n-1} [P(\varphi(\theta_k))x'(\theta_k^x) + Q(\varphi(\theta_k))y'(\theta_k^x) + R(\varphi(\theta_k))z'(\theta_k^x)](t_{k+1} - t_k), \qquad (*)$$

where $\varphi(\theta_k) = P_k$ , $k = 0, \ldots, n-1$, are the intermediate points of $\delta$.

Let us note the unit tangent vector at a current point of $\gamma$ by $\vec{C} = \dfrac{\vec{r}\,'}{\|\vec{r}\,'\|}$.

More exactly, if $M = \varphi(\theta)$, $\theta \in [a, b]$, then

$$\vec{C}(M) = \frac{x'(\theta)\vec{i} + y'(\theta)\vec{j} + z'(\theta)\vec{k}}{\sqrt{x'^2(\theta) + y'^2(\theta) + z'^2(\theta)}}.$$

Let us consider the scalar function $f = <\vec{F}, \vec{\zeta}>$, which has the integral sums of the first type (see remark 3 in §2)

$$S_{\gamma,\vec{F}}(\delta, \mathscr{S}) = \sum_{k=0}^{n-1} (f \circ \varphi)(\theta_k)\|\vec{r}\,'(\hat{\theta}_k)\|(t_{k+1} - t_k). \qquad (**)$$

By comparing the integral sums of $\vec{F}$ and $f$ , we naturally claim that the line integral of the second order of $\vec{F}$ reduces to the line integral of the first order of $f$. In fact, this relation is established by the following

**3.4. Theorem.** Under the above notations, if $\vec{F}$ is continuous on $\gamma$, then $\vec{F}$ is integrable on $\gamma$, and we have $\displaystyle\int_\gamma \vec{F}\, d\vec{r} = \int_\gamma f\, ds$ .

Proof. If $\vec{F}$ is continuous, then $f$ is continuous too, since $\vec{C}$ is continuous for smooth curves. Consequently, according to theorem 4 in §2, $f$ is integrable on $\gamma$. It remains to evaluate

$$| S_{\gamma,\vec{F}}(\delta, \mathscr{S}) - \int_\gamma f\,ds| \leq$$

$$\leq | S_{\gamma,\vec{F}}(\delta, \mathscr{S}) - S_{\gamma,f}(\delta, \mathscr{S})| + |S_{\gamma,f}(\delta, \mathscr{S}) - \int_\gamma f\,ds|.$$

The last modulus is less than $\frac{\varepsilon}{2}$ for $\|\delta\| < \eta_1$, hence it remains to find an upper bound of the other modulus. In fact, using (*) and (**) we obtain :

$$| S_{\gamma,\vec{F}}(\delta, \mathscr{S}) - S_{\gamma,f}(\delta, \mathscr{S})| \leq$$

$$\sum_{k=0}^{n-1} |\frac{P\circ\varphi}{\|\vec{r}\,'\|}(\theta_k)[x'(\theta_k^x)\|\vec{r}\,'(\theta_k)\| - x'(\theta_k)\,\|\vec{r}\,'(\widehat{\theta}_k)\|\,](t_{k+1} - t_k)| +$$

$$+ \sum_{k=0}^{n-1} |\frac{Q\circ\varphi}{\|\vec{r}\,'\|}(\theta_k)[y'(\theta_k^y)\|\vec{r}\,'(\theta_k)\| - y'(\theta_k)\,\|\vec{r}\,'(\widehat{\theta}_k)\|\,](t_{k+1} - t_k)| +$$

$$+ \sum_{k=0}^{n-1} |\frac{R\circ\varphi}{\|\vec{r}\,'\|}(\theta_k)[z'(\theta_k^z)\|\vec{r}\,'(\theta_k)\| - z'(\theta_k)\,\|\vec{r}\,'(\widehat{\theta}_k)\|\,](t_{k+1} - t_k)| .$$

Using the uniform continuity of the functions $P \circ \varphi$, $Q\circ \varphi$, $R\circ \varphi$, $\|\vec{r}\,'\|$ (which also is different from zero!), and $x'$, $y'$, $z'$ on $[a, b]$, this expression is also less than $\frac{\varepsilon}{2}$ for $\|\delta\| < \eta_2$ . ◇

**3.5. Corollary.** The line integral of the second order of a continuous function on "smooth curve " does not depend on the parameterization (up to sign, which is determined by the orientation!).

Proof. Because $\|\vec{C}\| = 1$, $f$ does not depend on parameterization, hence it remains to apply theorem 4 in §2, which expresses a similar property of the line integrals of the first type. ◇

**3.6. Corollary.** For any parameterization $\varphi : [a, b] \to \mathbb{R}^3$ of $\gamma$, we have:

$$\int_\gamma \vec{F}\,\mathrm{d}\vec{r} =$$

$$= \int_a^b [P(x(t), y(t), z(t))x'(t) + Q(x(t), y(t), z(t))y'(t) + R(x(t), y(t), z(t))z'(t)]\mathrm{d}t.$$

Proof. Using theorem 4 in §2, for $f = \vec{F}\,\vec{C}$, we obtain

$$\int_\gamma \vec{F}\,\mathrm{d}\vec{r} = \int_\gamma f\,ds = \int_a^b (f\circ\varphi)(t)\|\,\varphi'(t)\|\mathrm{d}t =$$

$$= \int_a^b ((\vec{F}\,\vec{r}\,') \circ \varphi)(t)\, \frac{1}{\|\vec{r}\,'(t)\|}\, \|\varphi'(t)\|\, \mathrm{d}t =$$

$$= \int_a^b [(P \circ \varphi)(t)x'(t) + (Q \circ \varphi)(t)y'(t) + (R \circ \varphi)(t)z'(t)]\, \mathrm{d}t,$$

where we remarked that $\|\vec{r}\,'(t)\| = \|\varphi'(t)\|$. ◇

The general properties of the line integral of the second type can be obtained from the similar properties of the line integral of the first type (formulated in theorem 5, §2).

**3.7. <u>Theorem</u>.** The line integral of the second order has the properties:

(i) Linearity relative to the functions:

$$\int_\gamma (\lambda\vec{F} + \mu\vec{G})\mathrm{d}\vec{r} = \lambda\int_\gamma \vec{F}\,\mathrm{d}\vec{r} + \mu\int_\gamma \vec{G}\,\mathrm{d}\vec{r}$$

(ii) Additivity relative to the union of curves

$$\int_{\gamma_1\cup\gamma_2} \vec{F}\,\mathrm{d}\vec{r} = \int_{\gamma_1} \vec{F}\,\mathrm{d}\vec{r} + \int_{\gamma_2} \vec{F}\,\mathrm{d}\vec{r}$$

(iii) Orientation relative to the sense on the curve $\int_\gamma \vec{F}\,\mathrm{d}\vec{r} = -\int_{\gamma^-} \vec{F}\,\mathrm{d}\vec{r}$.

<u>Proof.</u> Properties (i), (ii) are direct consequences of (i), (ii) of theorem 5, §2. Relative to (iii), it is necessary to remark that even if the line integral of the first type is the same on $\gamma$ and $\gamma^-$, function $f$ in the formula established in the above theorem 3.4 depends on the sense chosen on $\gamma$. In fact, if $\varphi : [a, b] \to \mathbb{R}^3$ is a parameterization of $\gamma$, then $\overline{C}(\varphi(\theta)) = -\overline{C}(\psi(t))$ at each $\varphi(\theta) = \psi(t) \in \gamma$. ◇

**3.8. <u>Remark.</u>** By calculating line integrals of the second type, we can see that sometimes the result *does not depend* on the curve but only on the endpoints (see problem 2). In practice this is an important case, for example, when the integral represents the work of a force, so it must be carefully analyzed. This property of the line integral will be studied in terms of "total differentials". More exactly, $\vec{F}\,\mathrm{d}\vec{r}$ is considered to be a *total differential* iff there exists a differentiable function $U : D \to \mathbb{R}$ such that

$$dU = \vec{F}\,\mathrm{d}\vec{r} = Pdx + Qdy + Rdz.$$

Alternatively, $\vec{F}\,\mathrm{d}\vec{r}$ is a total differential iff $\vec{F} = grad\ U$, i.e. $\vec{F}$ *derives from a potential.*

**3.9. <u>Theorem.</u>** (i) If $D \subseteq \mathbb{R}^3$ is an open set, and $U : D \to \mathbb{R}$ is a differentiable function such that $\vec{F} = grad\ U$, then for any smooth curve $\gamma \subset D$, of end-points $A$ and $B$, we have

$$\int_\gamma \vec{F}\,\mathrm{d}\vec{r} = U(B) - U(A),$$

i.e. the line integral of $\vec{F}$ is not depending on $\gamma$ .
(ii) Conversely, if $D \subseteq \mathbb{R}^3$ is an *open and connected* set, and $\vec{F} : D \to \mathbb{R}^3$ is a continuous vector function for which the line integral depends only on the end-points of the curves, then $\vec{F}\,\mathrm{d}\vec{r}$ is a total differential.
<u>Proof.</u> (i) If $\varphi : [a, b] \to \mathbb{R}^3$ is a parameterization of $\gamma$, and according to the hypothesis $P = \dfrac{\partial U}{\partial x}$, $Q = \dfrac{\partial U}{\partial y}$, $R = \dfrac{\partial U}{\partial z}$, then

$$\int_\gamma \vec{F}\,\mathrm{d}\vec{r} = \int_\gamma Pdx + Qdy + Rdz = \int_\gamma \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz =$$

$$= \int_a^b \Big[\frac{\partial U}{\partial x}(\varphi(t))\,x'(t) + \frac{\partial U}{\partial y}(\varphi(t))\,y'(t) + \frac{\partial U}{\partial z}(\varphi(t))\,z'(t)\Big]\,dt =$$

$$= \int_a^b (U \circ \varphi)'(t)\,dt = U(\varphi(b)) - U(\varphi(a)) = U(B) - U(A).$$

(ii) We have to construct $U$, for which $\vec{F} = grad\ U$. With this aim we fix $A = (x_0,y_0,z_0) \in D$, and we let $B = (x, y, z)$ free in $D$. Because $D$ is open and connected, it will also be connected by arcs, hence there exists a smooth curve $\gamma \subset D$ of end-points $A$ and $B$. Consequently, we may define a function $U : D \to \mathbb{R}$ by formula

$$U(x, y, z) = \int_A^B \vec{F}\,\mathrm{d}\vec{r},$$

where we mention only the points $A$ and $B$ because, by hypothesis, the considered line integral does not depend on the curve, which has these end-points. It remains to show that $\dfrac{\partial U}{\partial x} = P$, $\dfrac{\partial U}{\partial y} = Q$, $\dfrac{\partial U}{\partial z} = R$, at any point $B = (x, y, z) \in D$. In fact,

$$U(x + h, y, z) - U(x, y, z) = \int_{\gamma_h} \vec{F}\,\mathrm{d}\vec{r},$$

where $\gamma_h$ is any curve (in particular a straight segment) between $(x, y, z)$ and $(x + h, y, z)$ .

Using the parameterization $\varphi_h(t) = (x + th, y, z)$ of $\gamma_h$, we obtain

$$U(x + h, y, z) - U(x, y, z) = h\int_0^1 P(x + th, y, z)dt.$$

Applying the mean-value theorem to the last integral, it follows that there exists $\theta \in (0, 1)$ such that $\int_0^1 P(x + th, y, z)dt = P(x + \theta h, y, z)$, hence

$$\frac{U(x+h,y,z)-U(x,y,z)}{h} = P(x + \theta h, y, z).$$

Since $P$ is continuous (as a component of $\vec{F}$), it follows that there exists

$$\frac{\partial U}{\partial x}(x, y, z) = \lim_{h\to 0} \frac{U(x+h,y,z)-U(x,y,z)}{h} = P(x, y, z).$$

Similarly we evaluate the other partial derivative of $U$. ◇

**3.10. <u>Remark.</u>** (i) Beyond the existence of the potential $U$, the above theorem contains a formula, which gives $U$ concretely, namely

$$U(x, y, z) = \int_{(x_0,y_0,z_0)}^{(x,y,z)} Pdx + Qdy + Rdz.$$

More than this, because this integral is independent of the curve, we can chose it such that to obtain the most convenient calculation. In practice, it is frequently prefered a broken line

$\gamma = [(x_0, y_0, z_0), (x, y_0, z_0)] \cup [(x, y_0, z_0), (x, y, z_0)] \cup [(x, y, z_0), (x, y, z)]$, when the line integral reduces to three simple (Riemann) integrals, i.e.

$$U(x, y, z) = \int_{x_0}^{x} P(t, y_0, z_0)dt + \int_{y_0}^{y} Q(x, t, z_0)dt + \int_{z_0}^{z} R(x, y, t)dt.$$

This formula provides $U$ up to a constant which corresponds to the choice of $(x_0,y_0,z_0)$, and equals $U(x_0, y_0, z_0)$. A practical key of a correct calculation is the reduction of the "mixed" terms, which are evaluated at $(x, y_0, z_0)$, $(x, y, z_0)$, etc.

(ii) The above formulas for calculating $U$ can be considered as rules of determining a function when its differential is known; in other words this means finding *anti-derivatives* (or *primitives*) of a given function. Simple examples show that only particular triplets of functions $(P, Q, R)$ represent partial derivatives of a function $U$, so it is very important for practical purposes to know how to identify these cases.

**3.11. <u>Definition.</u>** We say that the field $\vec{F} \in C^1_{\mathbb{R}^3}(D)$ is *conservative* iff its components $P, Q, R$ satisfy the conditions

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \quad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$$

at each point of $D$. Instead of "conservative" many authors use the term "irotational" which derives from the notion of "rotation". More exactly, the *rotation* of $\vec{F} = (P, Q, R)$, noted $rot\vec{F}$, is defined as a vector formally expressed by the determinant

$$rot\ \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}.,$$

The assertion "$\vec{F}$ is conservative" really reduces to "$rot\ \vec{F} = 0$".

**3.12.** **Theorem.** Let $D \subseteq \mathbb{R}^3$ be an open and star-like set, and let $\vec{F} \in C^1_{\mathbb{R}^3}(D)$ be a vector field. A necessary and sufficient condition for $\vec{F}$ to derive from a potential is to be conservative.

We remind that a domain $D \subseteq \mathbb{R}^3$ is said to be *star-like* if there exists $M_0 \in D$ such that $\overline{M_0M} \subseteq D$ holds for all $M \in D$.

Proof. If $\vec{F} = grad\ U$ for some $U : D \to \mathbb{R}$, then $\vec{F}$ has the components $P = \dfrac{\partial U}{\partial x}$, $Q = \dfrac{\partial U}{\partial y}$, $R = \dfrac{\partial U}{\partial z}$. Because $P, Q, R \in C^1_{\mathbb{R}^3}(D)$, we can apply Schwartz' theorem (on mixed second order partial derivatives) to $U$, and so we easily see that $\vec{F}$ is conservative.

Conversely, let $\vec{F}$ be conservative on $D$. Since $D$ is star-like there exists $M_0 \in D$ such that for any other $M \in D$ we have $\overline{M_0M} \subset D$. A parameterization of this segment is

$$\varphi(t) = (x_0 + t(x - x_0), y_0 + t(y - y_0), z_0 + t(z - z_0)),\ t \in [0,1].$$

Let us define $U : D \to \mathbb{R}$, by

$$U(x, y, z) = \int_{\overline{M_0M}} \vec{F}\, d\vec{r}\, .$$

Using the parameterization φ in the formula established in corollary 3.6, we obtain

$$U(x, y, z) = \int_0^1 [(P \circ \varphi)(t)(x - x_0) + (Q \circ \varphi)(t)(y - y_0) + (R \circ \varphi)(t)(z - z_0)]dt$$

According to theorem V.1.5, concerning the derivation relative to a parameter in a definite integral, we have

$$\frac{\partial U}{\partial x}(x, y, z) =$$

$$= \int_0^1 [\frac{\partial P}{\partial x}(\varphi(t))t(x - x_0) + (P \circ \varphi)(t) + \frac{\partial Q}{\partial x}(\varphi(t))t(y - y_0) + \frac{\partial R}{\partial x}(\varphi(t))t(z - z_0)]dt$$

The hypothesis of being conservative allows us to express this integral only by the partial derivatives of $P$, i.e.

$$\frac{\partial U}{\partial x}(x, y, z) = \int_0^1 [t(P \circ \varphi)'(t) + (P \circ \varphi)(t)]dt =$$

$$= \int_0^1 [t(P \circ \varphi)]'(t)dt = (P \circ \varphi)(1) = P(x, y, z).$$

Consequently, for any $(x, y, z) \in D$ we have $\frac{\partial U}{\partial x}(x, y, z) = P(x, y, z)$. Similarly, we prove that $\frac{\partial U}{\partial y} = Q$, and $\frac{\partial U}{\partial z} = R$. ◇

Simple examples show that the above condition on $D$ to be star-like is essential for a conservative field to derive from a potential.

**3.13.** **Example.** On the open (but not star-like) set $D = \mathbb{R}^2 \setminus \{(0,0)\}$ we consider the field $\vec{F}(x, y) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$. Obviously, we have $\vec{F} \in C^1_{\mathbb{R}^3}(D)$, and $\frac{\partial P}{\partial y} = \frac{\partial R}{\partial x}$, hence $\vec{F}$ is conservative on $D$. Now, let $\gamma$ be the unit circle in the plane traced counter clockwise. Since

$$\int_\gamma \vec{F}\, d\vec{r} = 2\pi \neq 0$$

it is clear that $\vec{F}$ can not derive from a potential.

**3.14.** **Conclusion.** In practice, when we have to calculate a line integral of the second type, it is useful primarily to check whether the corresponding vector field is conservative or not. If it isn't conservative we must find a parameterization of the curve and apply the most general formula (as in corollary 3.6). If the field is conservative (and the domain is star-like!), we apply the formula in theorem 3.9 (*i*), when $U$ may be obtained as in remark 10, (*i*) .

Finally, we mention another application of the line integral of the second type (in addition to the work of a force).

**3.15.** **Proposition.** Let $\gamma$ be a simple, smooth and closed contour, traced one time counter-clockwise, and having the property that any parallel to the *ox* and to *oy* axis meets the curve at most twice. Then the area bounded by $\gamma$ is expressed by

$$\mathcal{A} = \frac{1}{2}\int_\gamma xdy - ydx.$$

Proof. We can consider $\gamma = \gamma_1 \cup \gamma_2$ as in Fig. VI.3.2 (*a*), and alternatively $\gamma = \gamma_3 \cup \gamma_4$ as in Fig. VI.3.2 (*b*).

By interpreting $\int_{\gamma_1} y\,dx$ and $\int_{\gamma_2} y\,dx$ like areas of sub-graphs, we obtain

$$\mathcal{A} = -\int_\gamma y\,dx.$$

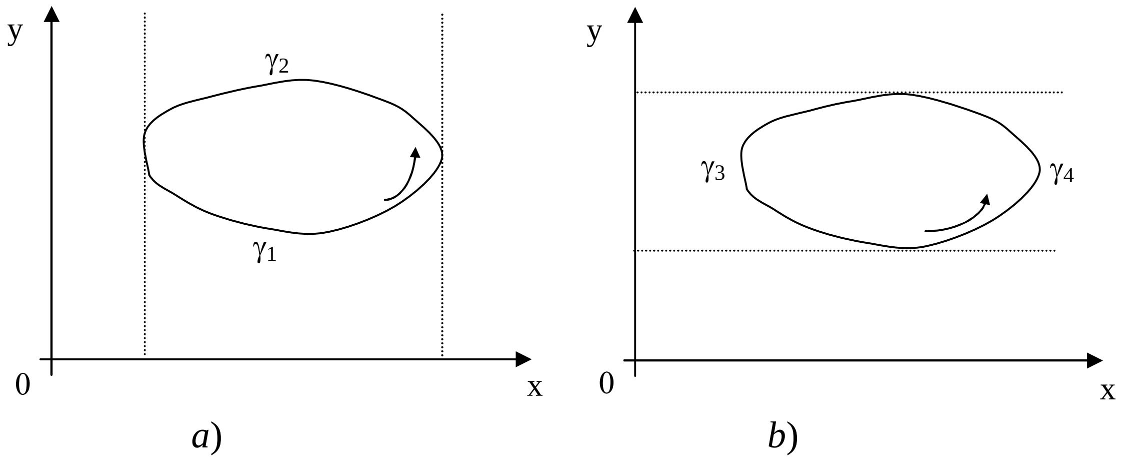



Fig. VI.3.2.

Similarly,

$$\mathscr{A} = \int_{\gamma} x\, dy.$$

It remains to add the two expressions of $\mathscr{A}$. ◇

This formula of $\mathscr{A}$ is be a particular case of the Green-Riemann formula (see later VII.2.21 and 22). There exist many similar formulas of the area, which involve non-Euclidean coordinates. In particular:

**3.16. <u>Example</u>.** Let us say we need the formula of the area of a plane domain $D$, which is bounded by a closed curve, explicitly expressed in polar coordinates by the equation $r = \varphi(\theta)$, where $\varphi : [\theta_1, \theta_2] \to \mathbb{R}^+$. In this case we have to evaluate $\mathscr{A} = \dfrac{1}{2}\displaystyle\int_{\theta_1}^{\theta_2} \varphi^2(\theta)d\theta$.

In particular, the domain contained inside Bernoulli's lemniscate (of equation $r^2 = a^2 \cos 2\theta$) has the area $\mathscr{A} = a^2$.

## PROBLEMS § VI.3.

**1.** Evaluate the work of the forces

$$\vec{F} = x\,\vec{i} + y\,\vec{j} \quad \text{and} \quad \vec{G} = y\,\vec{i} - x\,\vec{j}$$

in the process of moving a material point along an ellipse of half-axes $a$ and $b$ in the $xoy$ plane.

Hint. We calculate $\int_\gamma \vec{F}\,\mathrm{d}\vec{r}$ and $\int_\gamma \vec{G}\,\mathrm{d}\vec{r}$, where $\gamma$ has the parametric equations $x = a\cos t$, $y = b\sin t$, $t \in [0, 2\pi)$.

**2.** Calculate $\int\limits_{\gamma|_{OA}} 2xydx \pm x^2\,dy$, where $O = (0, 0)$, $A = (2, 1)$, for different arcs $\gamma|_{OA}$ in the plane (straight line, parabolas, broken lines) (of end-points $O$ and $A$).

**3.** Find the work of the force $\vec{F}\,(x, y, z) = (y - z, z - x, x - y)$ by moving a point along the screw line $\gamma$ of parameterization $x = a\cos t$, $y = b\sin t$, $z = bt$, $t \in [0, 2\pi)$.

Solution. $w = \int_\gamma \vec{F}d\vec{r} = -2\,\pi a(a + b)$.

**4.** Calculate the integral $\int\limits_\gamma \dfrac{xy(ydx - xdy)}{x^2 + y^2}$, where $\gamma$ is the right-hand loop of the lemniscate $r^2 = a^2\cos 2\alpha$, traced counter-clockwise.

Hint. A parameterization of $\gamma$, in polar coordinates, is:

$$\begin{cases} x = r\cos\alpha = a\cos\alpha\sqrt{\cos 2\alpha} \\ y = r\sin\alpha = a\sin\alpha\sqrt{\cos 2\alpha} \end{cases}, \quad \alpha \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right].$$

The integral is null.

**5.** Find the anti-derivative $U$ if the differential is:

(i) $dU = (2x + 3y)dx + (3x - 4y)dy$

(ii) $dU = e^{x-y}\,[(1 + x + y)dx + (1 - x - y)dy]$

(iii) $dU = xdx + ydy$

(iv) $dU = xdy + ydx$

(v) $dU = \dfrac{ydx - xdy}{y^2}$

(vi) $dU = y^2zdx + (2xyz + 1)dy + xy^2dz$

(vii) $dU = \dfrac{2x}{x^2 + y^2}dx + \dfrac{2y}{x^2 + y^2}dy + 2zdz$.

Hint. Verify that the corresponding field is conservative (i.e. the problem is correctly formulated), identify the domain and calculate $U$ using the formula in remark 3.10, (i).

**6.** Find the anti-derivatives of the integrands and calculate:

(i) $\displaystyle\int_{(-1,-1)}^{(1,2)} (x^4 + 4xy^3)\, dx + (6x^2y^2 - 5y^4)\, dy$

(ii) $\displaystyle\int_{(1,0)}^{(0,1)} \frac{(x+2y)dx + ydy}{(x+y)^2}$, where $\gamma$ does not intersect the straight line of equation $y = -x$

(iii) $\displaystyle\int_{(0,0,0)}^{(1,1,1)} \left(\frac{x}{\sqrt{x^2+y^2}} + y\right)dx + \left(\frac{y}{\sqrt{x^2+y^2}} + x\right)dy$.

**7.** Evaluate the line integrals of the total differentials:

(i) $\displaystyle\int_{(1,1,1)}^{(2,2,2)} yz\, dx + zx\, dy + xy\, dz$

(ii) $\displaystyle\int_{(0,0,0)}^{(1,1,1)} \frac{xdx + ydy + zdz}{\sqrt{x^2+y^2+z^2}}$

(iii) $\displaystyle\int_{(1,1,1)}^{(x,y,\frac{1}{xy})} \frac{yzdx + zxdy + xydz}{xyz}$, where the integration curve is situated in the first octant.

**8.** Find the work of the Newtonian force $\vec{F} = -\frac{\mu}{r^3}\cdot\vec{r}$, which is necessary to move a material point from $A(x_1, y_1, z_1)$ to $B(x_2, y_2, z_2)$ along an arbitrary curve $\gamma$ of these endpoints, such that $(0, 0, 0) \notin \gamma$.

Hint. $-\vec{F}$ derives from the scalar potential

$$U(x,y,z) = \int_1^x \frac{\mu\, t\, dt}{(t^2+2)^{3/2}} + \int_1^y \frac{\mu\, t\, dt}{(x^2+t^2+1)^{3/2}} + \int_1^z \frac{\mu\, t\, dt}{(x^2+y^2+t^2)^{3/2}}.$$

Consequently, $W = \int_A^B \vec{F}\cdot d\vec{r} = U(x,y,z)\Big|_{(x_1,y_1,z_1)}^{(x_2,y_2,z_2)}$, where $U = \frac{\mu}{r} - \frac{\mu}{\sqrt{3}}$.

**9.** Show that if $f : \mathbb{R}^+ \to \mathbb{R}$ is continuous, and $\gamma$ is a closed piecewise smooth contour, then $\int_\gamma f(x^2 + y^2)(x\,dx + y\,dy) = 0$.

Hint. Consider $\Phi(r) = \int_0^r f(t)dt$ and $V(x, y) = \frac{1}{2}\Phi(x^2 + y^2)$, such that the integral becomes $dV$.

**10.** A circle of radius $r$ is rolling without sliding along a fixed circle of radius $R$ and outside it. Assuming that $\frac{R}{r}$ is an integer, find the area bounded by the epicycloid (hypocycloid) determined by some point of the moving circle. Analyze the particular case of the cardioid, (where $R = r$), and asteroid, (when $R = 4r$).
Hint. A parameterization of the epicycloid is

$$x = (R + r)\cos t - r\cos\frac{R+r}{2}t,\ \ y = (R + r)\sin t - r\sin\frac{R+r}{2}t,$$

where $t \in [0, 2\pi)$ is the angle between two radiuses of the fixed circle, one corresponding to the starting common point, and the other to an arbitrary current point. The parameterization of the hypocycloid is obtained by replacing $r$ by $-r$. Answer: $\pi(R \pm r)(R \pm 2r)$.

**11.** Evaluate $I = \int_\Gamma \frac{xdx + ydy + zdz}{(x^2 + y^2 + z^2)^{3/2}}$ where $\Gamma$ is a smooth curve of endpoints $(1, 0, 0)$ and $(0, 1, 0)$.
Hint. $\vec{V} = (x^2 + y^2 + z^2)^{-3/2}(x\vec{i} + y\vec{j} + z\vec{k})$ derives from a potential $U$, hence $I = U(B) - U(A)$.

# CHAPTER VII. MULTIPLE INTEGRALS

In this chapter we'll extend the notion of integral defined on an interval $I \subseteq \mathbb{R}$, and that of line integral along a curve, by considering integrals on domains in $\mathbb{R}^2$, $\mathbb{R}^3$ and generally in $\mathbb{R}^p$. These are called "multiple integrals" because of the higher dimension of the considered domains. The whole theory is based on the notions of "area" and "volume" which extend the notion of "length". Because all these notions are particular cases of "measures", for the beginning we have to clarify some topics concerning the Jordan's measure in $\mathbb{R}^p$, $p \in \mathbb{N}^*$.

## § VII.1. JORDAN'S MEASURE

It is well known that in the process of calculating areas and volumes, we start out with simple figures like rectangles and rectangular parallelepipeds, which are later used for approximating other figures (a significant example is the area of a sub-graph). This method can be unitarily applied in order to measure bodies in $\mathbb{R}^p$, for arbitrary $p \in \mathbb{N}^*$.

**1.1. <u>Definition.</u>** If $P = [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_p, b_p]$ is a closed rectangular *parallelepiped* (also called $p$–dimensional interval, or *"paralleloid"*), then the number

$$v(P) = \prod_{k=1}^{p} |b_k - a_k|$$

is called *$p$–volume* of $P$, or *measure* of $P$.

Any finite union of such closed rectangular parallelepipeds, each pair of them having no common interior point, is named *elementary body*. The *$p$–volume* (or *measure*) of an elementary body is the sum of the $p$–volumes of all parallelepipeds which form the body.

**1.2. <u>Remark.</u>** (i) The idea of considering <u>finite</u> families of parallelepipeds in the so called elementary body is specific to the measure theory in Jordan's sense. The alternative is the Lebesgue's point of view of taking *countable* families of parallelepipeds in the elementary bodies. We will develop here the Jordan's measure theory because it is simpler, and it is sufficient for studying the Riemann multiple integrals. However, we mention that the simplicity of Jordan's measure is counter-balanced by some disadvantages (for example, see later the notion of measurable set).

(ii) The union of two elementary bodies is an elementary body, and the same for the adherence of the difference (not for the difference itself).

(iii) The measure of an elementary body does not depend on its decomposition into rectangular parallelepipeds; there are still problems when the sides are no longer parallel to the axis.

**1.3. <u>Definition.</u>** Let $A \subset \mathbb{R}^p$ be an arbitrary bounded set, and let P', P'' denote elementary bodies in $\mathbb{R}^p$. Then the number

$$\mu_i(A) = \sup\ \{v(P') : P' \subseteq A\}$$

is called *Jordan's interior* (or "internal") *measure* of $A$, and

$$\mu_e(A) = \inf\ \{v(P'') : P'' \supseteq A\}$$

is called *Jordan's exterior* (or "external") *measure* of $A$. These notions making sense since $A$ is bounded.

If $\mu_i(A) = \mu_e(A)$ we say that $A$ is *measurable* in Jordan's sense, and the common value, denoted $\mu(A) = \mu_i(A) = \mu_e(A)$ is called the *Jordan's measure of A.*

**1.4. <u>Examples.</u>** (i) The set $A = \{\frac{1}{n} : n \in \mathbb{N}^*\}$ is measurable in $\mathbb{R}$, and $\mu(A) = 0$. There are still countable sets (e.g. $\mathbb{N}$, $\mathbb{Q} \cap [0,1]$, etc.), which are not measurable, hence this property depends on the position of the terms (unlike the Lebesgue's measure, which is null for any countable set).

(ii) The elementary bodies and their interiors are measurable sets, and we always have $\mu(B) = v(B) = \mu(\overset{\bullet}{B})$.

(iii) If $A$ and $B$ are measurable sets in $\mathbb{R}^p$, then $A \cup B$, $A \cap B$, $A \setminus B$ are also measurable (for more details see Theorem 1.6 below).

In order to evaluate measures, the following lemma is helpful:

**1.5. <u>Lemma.</u>** Let $A$ be a bounded set in $\mathbb{R}^p$. For $A$ to be measurable is necessary and sufficient that for each $\varepsilon > 0$ there exist some elementary bodies $P' \subseteq A$ and $P'' \supseteq A$ such that $\mu(P'') - \mu(P') < \varepsilon$.

<u>Proof.</u> Because always $\mu(P'') \geq \mu(P')$, it enough to express the condition

$$\sup\ \{\mu(P') : P' \subseteq A\} = \inf\ \{\mu(P'') : P'' \supseteq A\}$$

in terms of $\varepsilon > 0$. ◇

Now we can establish some properties of the Jordan's measure:

**1.6. <u>Theorem.</u>** If $A, B \subset \mathbb{R}^p$ are measurable sets, then:

(i) $A \subseteq B$ implies $0 \leq \mu(A) \leq \mu(B)$

(ii) If $A \cap B$ contains no paralleloid of non-null $p$–volume, then

$$\mu(A \cup B) = \mu(A) + \mu(B)$$

(iii) If $A \subseteq B$ then $\mu(B \setminus A) = \mu(B) - \mu(A)$ .

<u>Proof.</u> (*i*) Is obvious. (ii) is based on the fact that for two paralleloids P, Q with no common interior points we have $\mu(P \cup Q) = \mu(P) + \mu(Q)$. (iii) We may apply (ii) to $B = A \cup (B \setminus A)$. ◇

Because the sets with null measure play an important role in measure theory, we distinguish them by a special term:

**1.7. Definition.** We say that a set $A \subset \mathbb{R}^{\mathbf{p}}$ is *negligible* iff it is measurable and $\mu(A) = 0$

**1.8. Remarks.** (i) The finite sets are negligible. The finite unions of negligible sets are also negligible. Any subset of a negligible set is negligible.

(ii) Since we always have $\mu_i(A) \leq \mu_e(A)$, it follows that $A$ is negligible iff $\mu_e(A) = 0$. In other terms, $A$ is negligible iff for every $\varepsilon > 0$ there exists an elementary body $B$, such that $A \subseteq B$ and $v(B) < \varepsilon$.

(iii) If $A$ is bounded in $\mathbb{R}^{\mathbf{p}}$, and $p < q$, then $A$ is negligible sets in $\mathbb{R}^{\mathbf{q}}$. In particular, the segment $[a, b] \subset \mathbb{R}$ is negligible in $\mathbb{R}^2$. More generally, the fact that the smooth curves and surfaces are negligible in $\mathbb{R}^3$ is a consequence of the following theorem:

**1.9. Theorem.** Let $A \subset \mathbb{R}^{\mathbf{p}}$ be bounded. If $f : A \to \mathbb{R}^{\mathbf{q}}$, where $p < q$, is Lipschitzean (i.e. there exists $c > 0$ such that $\|f(x) - f(y)\| < c\, \|x - y\|$ for all $x, y \in A$), then $f(A)$ is negligible in $\mathbb{R}^{\mathbf{q}}$.

Proof. Let $K \subset \mathbb{R}^{\mathbf{p}}$ be a $p$–cube of side $h$ such that $A \subseteq K$. By dividing each side into $n$ equal parts, the cube breaks up into $n^p$ cubes of side $\frac{h}{n}$. We claim that if $\omega$ is such a small cube, then $f(A\cap \omega)$ is contained in a cube of side $2c^p\frac{h}{n}$ in $\mathbb{R}^{\mathbf{q}}$. In fact, if $A\cap \omega = \varnothing$ or consists of a single point, the assertion is obvious. If $A\cap \omega$ consists of more than two points, then we fix $a \in A\cap \omega$, and for any other $x \in A\cap \omega$, we obtain

$$\|f(x) - f(a)\| \leq c\|x - a\| \leq cp\frac{h}{n},$$

i.e. $f(A \cap \omega) \subseteq S\left( f(a), cp\frac{h}{n}\right)$. It is sufficient to remark that this sphere is included in a cube of side $2cp\frac{h}{n}$.

If P is the union of all the cubes which contain sets of the form $f(A \cap \omega)$, then $f(A) \subseteq$ P, and

$$\mu(\mathrm{f(A)}) \leq \mu(\mathrm{P}) \leq n^p \left(\frac{2cph}{n}\right)^q = (2cph)^q \frac{1}{n^{q-p}}.$$

From $q > p$ it follows that $\lim\limits_{n\to\infty} \frac{1}{n^{q-p}} = 0$, hence $f(A)$ is negligible. ◇

The following theorem shows that the negligible sets are very useful in establishing the measurability of other sets.

**1.10. <u>Theorem.</u>** Let $A \subset \mathbb{R}^p$ be a bounded set. A necessary and sufficient condition for $A$ to be measurable is that Fr$A$ be negligible (in the sense of the Jordan's measure).

<u>Proof.</u> Let $A$ be measurable. Then for any $\varepsilon > 0$ there exists the elementary bodies P, Q such that $P \subseteq A \subseteq Q$ and $\mu(Q) - \mu(P) < \varepsilon$. Because $FrA \subseteq Q \setminus \overset{\circ}{P}$ and $\mu(P) = \mu(\overset{\circ}{P})$, we obtain $0 \leq \mu_e(Fr\,(A)) \leq \mu(Q \setminus \overset{\circ}{P}) = \mu(Q) - \mu(P) < \varepsilon$, which shows that Fr$A$ is negligible.

Conversely, let us suppose that Fr$A$ is negligible, i.e. for any $\varepsilon > 0$ there exists an elementary body $B$ such that Fr$A \subset \overset{\circ}{B}$, and $\mu(B) < \varepsilon$. It is easy to see that $A \setminus B$ is open and Fr$(A \setminus B) \subset$ Fr$B$. Let us note $P = \overline{A \setminus B}$, $Q = \overline{A \cup B}$. We claim that:

(i) P is an elementary body,
(ii) Q is an elementary body too,
(iii) $P \subseteq A \subseteq Q$, and
(iv) $\mu(Q) - \mu(P) < \varepsilon$.

These properties are sufficient to conclude that $A$ is measurable.

In fact, to prove *(i)* we remark that since $A \setminus B$ is open, for each $x \in A \setminus B$ there exists a paralleloid $P_x$ such that $x \in \overset{\circ}{P_x} \subseteq A \setminus B$.

Let us remark that $Fr\,(A \setminus B) \subset \overset{\circ}{A}$. In fact, on the contrary case, if $x \in Fr\,(A \setminus B)$ and $x \in Fr\,A$, then we deduce that for any neighborhood $V$ of $x$ we have $V \cap (A \cap \complement B) \neq 0$. In addition, $V \subset \overset{\circ}{B}$ holds for some of these neighborhoods (since $Fr\,A \subset \overset{\circ}{B}$), which is impossible. Consequently, for any $y \in Fr\,(A \setminus B)$ there exists a paralleloid $P_y$ such that $y \in \overset{\circ}{P_y} \subset A$. In conclusion, the family $\{\overset{\circ}{P_x} : x \in A \setminus B\} \cup \{\overset{\circ}{P_y} : y \in F_n(A \setminus B)\} \cup \{\overset{\circ}{B}\}$ is an open cover of $\overline{A}$. By hypothesis $A$ is bounded, hence $\overline{A}$ is compact, so there exists a finite subfamily

$$\{\overset{\circ}{P}_{x_i} : i = 1, ..., n\} \cup \{\overset{\circ}{P}_{y_j} : j = 1, ..., m\} \cup \{\overset{\circ}{B}\}$$

which also covers $\overline{A}$. In particular, removing $\overset{\circ}{B}$, this subfamily covers $P = \overline{A \setminus B}$, so it remains to modify the paralleloids of this cover such that $P$ to appear as an elementary body.

(ii) is immediate if we note that $Q = P \cup B$.

Similarly, in (iii), $A \subseteq Q$ is obvious, and $P \subseteq A$ is based on the fact that $Fr\,(A \setminus B) \subset A$.

Finally, for (*iv*) we evaluate $\mu(Q) - \mu(P) = \mu(\overline{(A \setminus B) \cup B}) - \mu(P) =$ $= \mu(P \cup B) - \mu(P) = \mu(B) < \varepsilon$. ◇

The last two theorems have useful consequences in the study of the measurability. For example:

**1.11. <u>Corollary.</u>** Let $A$ be a bounded (in particular compact) set in $\mathbb{R}^{\mathbf{p}}$, $p \in \mathbb{N}^*$. If *Fr* ($A$) consists of a finite number of smooth images of at most ($p$–1)–dimensional measurable sets, then $A$ is measurable.

<u>Proof.</u> Smooth functions are Lipschitzean, hence, according to theorem 3.9, *Fr* ($A$)is negligible. The rest is said by theorem 3.10. ◇

**1.12. <u>Examples.</u>** The $p$–dimensional ball $S(x_0,r)$ is measurable since its boundary is smooth. Similarly, any bounded polyhedron is measurable because its boundary consists of a finite number of ($p$–1)–dimensional flat surfaces. In particular, the parallelepipeds having faces non-parallel to the axes are measurable too. Evaluating their measure, as well as the preservation of the measure under isometries and other transformations, remain more complicated problems, which will not be studied here.

**PROBLEMS § VII.1.**

**1.** Show that the sub-graph of any bounded and increasing function $f : [a, b] \to \mathbb{R}$ is measurable (in Jordan's sense).

Hint. To any two elementary bodies P', P'' for which P'$\subseteq$ *sub-graph* $f \subseteq$ P'' there correspond two divisions δ' and δ'' of $[a, b]$ such that for

$$\delta = \delta' \cup \delta'' = \{a = x_0 < x_1 < ... < x_n = b\}$$

we have

$$v(P') \leq \sum_{k=0}^{n-1} f(x_k)(x_{k+1} - x_k) \leq \sum_{k=0}^{n-1} f(x_{k+1})(x_{k+1} - x_k) \leq v(P'').$$

For sufficiently fine divisions δ, when $||\delta|| < \eta$, the difference

$$\nu(P'') - \nu(P') = \sum_{k=0}^{n-1} [f(x_{k+1}) - f(x_k)](x_{k+1} - x_k)$$

will be arbitrary small.

**2.** Show that even though the function $f : [0,1] \to \mathbb{R}^2$,

$$f(t) = \begin{cases} (t, \sin\frac{1}{t}) & \text{if } t \in (0,1] \\ (0,0) & \text{if } t = 0 \end{cases}$$

is not Lipschitzean, the image $f([0,1])$is negligible in $\mathbb{R}^2$.

Hint. $||f(t_1) - f(t_2)|| \geq 1$ is possible for arbitrary closed $t', t'' \in [0, 1]$. For any $\varepsilon > 0$ there exists $\eta > 0$ such that $f([0, \eta])$ be included in a rectangle of area less than ε. The remaining $f([\eta, 1])$ is negligible according to theorem 3.9.

**3.** Compare the measures of $I = (0, 1) \subset \mathbb{R}$ and $f(I) \subset \mathbb{R}^2$, where the function $f : I \to \mathbb{R}^2$, is defined by $f(0.c_1c_2c_3c_4...) = (0.c_1c_3..., 0c_2c_4...)$. Are these non-negligible simple curves in $\mathbb{R}^p$, $p \geq 2$ ?

Hint. $f(I) = (0, 1) \times (0, 1)$, and $f$ is 1 : 1. However, if $a$ denotes the measure (area) in $\mathbb{R}^2$, we have $a(I) = 0$ and $a(f(I)) = 1$. Take $\gamma = f(I)$.

**4.** Study the measurability (in Jordan's sense) of the following sets in the plane:

$A = \{(x, y) \in [0, 1] \times [0, 1] : x, y \in \mathbb{Q}\}$

$B = \{(x, y) \in [0, 1] \times [0, 1] : x, y \in \mathbb{R} \setminus \mathbb{Q}\}$

$C = A \cup B$, and $\overline{C}$.

Answer. Only $\overline{C}$ is measurable.

## § VII.2. MULTIPLE INTEGRALS

The notion of multiple integral will be considered for an arbitrary dimension of the space $p \geq 2$, with particular stress on the cases $p = 2$ and $p = 3$ of the "double" and "triple" integrals. Starting out with some practical problems, we discuss the methods based on Darboux and Riemann sums.

**2.1. <u>Practical problems.</u>** (i) Let $D$ be a compact domain of the plane, bounded by a piecewise smooth curve $\gamma$, and let $f : D \to \mathbb{R}$ be a bounded function. If we have to calculate the volume of the sub-graph of $f$ in $\mathbb{R}^3$**,** we naturally divide the set $D$ into measurable sub-domains $D_k$ , $k = 1, ..., n$, of areas $a(D_k)$, and we approximate the asked volume by sums of the form

$$\sum_{k=1}^{n} (\inf [f(D_k)]) \cdot a(D_k),$$

$$\sum_{k=1}^{n} (\sup [f(D_k)]) \cdot a(D_k), \text{ or}$$

$$\sum_{k=1}^{n} [f(\xi_k)] \cdot a(D_k), \text{ where } \xi_k \in D_k.$$

In particular, the sub-domains $D_k$ can be rectangles, which constitute elementary bodies used in the process of obtaining the internal and external measure of $D$.

(ii) Let $D \subset \mathbb{R}^3$ be a compact set bounded by a piecewise smooth surface. If $D$ represents a physical body of density $f$, then the mass of $D$ may be approximated by sums of the form $\sum_{k=1}^{n} f(\xi_k)v(D_k)$, where $\xi_k \in D_k$, and $v(D_k)$ is the volume of $D_k$. Usually $D_k$ are parallelepipeds with no common interior points, in finite number, included in $D$.

The above sums suggest how to define the integral sums in the case of the multiple integrals, but first we must specify some terms:

**2.2. <u>Terminology and notations.</u>** The closure of a domain (open and connected set) is called *closed domain*. A bounded closed domain in $\mathbb{R}^p$, $p \in \mathbb{N}^*$ is called *compact domain*. If $D \subset \mathbb{R}^p$ is a measurable (in Jordan's sense) compact domain (briefly *m.c.d.*), then any finite family of sets, $\delta = \{D_1, D_2, ...., D_n\}$, which satisfies the conditions:

(i) each $D_k$ , $k = 1, ..., n$ is a *m.c.d.*,

(ii) $D = \bigcup_{k=1}^{n} D_k$ ,

(iii) $\overset{\circ}{D_k} \cap \overset{\circ}{D_l} = \emptyset$ whenever $k \neq l$, is called *division* of $D$. By *norm* of division $\delta$ we understand the number $\|\delta\| = max\{d(D_k) : k = 1, ..., n\}$, where $d(D_k) = \sup \{\|x - y\| : x, y \in D_k\}$ is the *diameter* of $D_k$. Division $\delta''$ is said to be *finer* than $\delta'$ iff for any $D' \in \delta'$ there exists $D'' \in \delta''$ such that $D' \supseteq D''$. If $\delta'$ and $\delta''$ are divisions of $D$, then

$$\delta = \{D' \cap D'' : D' \in \delta' \text{ and } D'' \in \delta''\}$$

is called *supremum* of $\delta'$ and $\delta''$ and it is denoted by $\delta = \delta' \vee \delta''$. The Jordan measure on $\mathbb{R}^p$ will be denoted by $\mu$.

**2.3. <u>The construction of the integral sums.</u>** Let $D \subset \mathbb{R}^p$ be a *m.c.d.*, let $f : D \to \mathbb{R}$ be bounded, and let $\delta = \{D_1, D_2, ...., D_n\}$ be a division of $D$. For each $k = 1, ..., n$ we note $m_k = \inf f(D_k)$, and $M_k = \sup f(D_k)$. The sum

$$s_f(\delta) = \sum_{k=1}^{n} m_k \, \mu(D_k)$$

is called *Darboux inferior sum*, while

$$S_f(\delta) = \sum_{k=1}^{n} M_k \, \mu(D_k)$$

is called *Darboux superior sum.*

If $\mathscr{S} = \{\xi_k : k = 1, ..., n\}$, where $\xi_k \in D_k$ is a system of *intermediate* points, then the sum

$$\sigma_f(\delta, \mathscr{S}) = \sum_{k=1}^{n} f(\xi_k) \, \mu(D_k)$$

is called *Riemannian sum*.

Using these sums we define the "multiple" integrals:

**2.4. <u>Definition.</u>** The number $\underline{I} = \sup\limits_{\delta} s_f(\delta)$ ($\overline{I} = \inf\limits_{\delta} S_f(\delta)$) is called *Darboux inferior* (*superior*) *integral* of $f$ on $D$. We say that $f$ is *Darboux integrable* on $D$ iff $\underline{I} = \overline{I}$. In such a case , the common value is denoted by

$$I = \underline{I} = \overline{I} = \int_D f d\mu$$

and is called *the multiple integral of f on D* (in Darboux' sense).

We say that $f$ is *integrable in Riemann's sense on D* iff there exists

$$\mathrm{L} = \lim_{\|\delta\| \to 0} \sigma_f(\delta, \mathscr{S})$$

and this limit is independent of the sequence of the divisions (with $\|\delta\| \to 0$) and of the systems of intermediate points. If so, L is named *the Riemann multiple integral of f on D*, and it is also denoted

$$\mathrm{L} = \int_D f d\mu$$

This coincidence of notations is based on the following:

**2.5. <u>Theorem.</u>** The following assertions are equivalent:
(i) $f$ is integrable on $D$ in Darboux' sense
(ii) for any $\varepsilon > 0$ there exists a division $\delta$ such that $S_f(\delta) - s_f(\delta) < \varepsilon$
(iii) $f$ is integrable on $D$ in Riemann's sense.

The proof is the same as for simple integrals on $\mathbb{R}$, and it will be omitted. In particular, assertion (i) $\Leftrightarrow$ (ii) represents the well known Darboux criterion of integrability.

**2.6. <u>Particular cases.</u>** If $p = 1$, and $D = [a, b] \subset \mathbb{R}$ we recognize that

$$\int_D f d\mu = \int_a^b f(x)dx.$$

In the case $p = 2$ we usually note

$$\int_D f d\mu = \iint_D f(x, y)dxdy$$

and we call it *double integral* of $f$ on $D$.

Similarly, when $p = 3$, we note

$$\int_D f d\mu = \iiint_D f(x, y, z)dxdydz$$

and we name it *triple integral* of $f$ on $D$ .

Even for $p$ greater than 3 the multiple integral is sometimes written in the form $\int_D f d\mu = \int \dots \int_D f d\mu = \int \dots \int_D f(x_1, \dots, x_p)dx_1 \dots dx_p$.

Using the previous theorem we obtain an important class of integrable functions:

**2.7. <u>Theorem.</u>** Let $D \subset \mathbb{R}^p$ be a <u>m.c.d.</u> , and let $f : D \to \mathbb{R}$ be a bounded function. If the set $\Delta = \{x \in D : f$ is discontinuous at $x\}$ is negligible, then $f$ is integrable on $D$ .

<u>Proof.</u> The case $\mu(D) = 0$ is trivial, so that we'll consider $\mu(D) > 0$. For the beginning we prove the theorem under the hypothesis $\Delta = \emptyset$, when $f$ (continuous) on $D$ (compact) is uniformly continuous, i.e. for any $\varepsilon > 0$ there exists $\eta > 0$ such that $\|x' - x''\| < \eta$ implies $|f(x') - f(x'')| < \frac{\varepsilon}{\mu(D)}$.

Now, if $\delta = \{D_1, D_2, \dots, D_n\}$ is a division of $D$, for which $\|\delta\| < \eta$, we obtain

$$S_f(\delta) - s_f(\delta) = \sum_{k=1}^{n} (M_k - m_k)\, \mu(D_k) = \sum_{k=1}^{n} [\, f(x'_k) - f(x''_k)]\, \mu(D_k) <$$

$$< \frac{\varepsilon}{\mu(D)} \sum_{k=1}^{n} \mu(D_k) = \varepsilon,$$

where the existence of $x'_k, x''_k \in D_k$ such that $f(x'_k) = M_k$ and $f(x''_k) = m_k$ for every $k = 1, ..., n$, is assured by the continuity of $f$ on the compacts $D_k$. Consequently, according to theorem 2.5, $f$ is integrable on $D$.

Finally, let us consider the general case when $\Delta \neq \emptyset$. Since $\mu(D) = 0$, for any $\varepsilon > 0$, there exists an elementary body $B$, such that $\Delta \subset \overset{\circ}{B}$ and $\mu(B) < \dfrac{\varepsilon}{4M}$, where $M = \sup \{|f(x)| : x \in D\}$. The set $D \setminus \overset{\circ}{B}$ is a *m.c.d.* too, on which $f$ is continuous, hence, as before, $f$ is integrable on $D \setminus \overset{\circ}{B}$. In other terms, there exists a division $\tilde{\delta} = \{D_1, D_2, ..., D_n\}$ of $D \setminus \overset{\circ}{B}$ such that $S_f(\tilde{\delta}) - s_f(\tilde{\delta}) < \dfrac{\varepsilon}{2}$.

Of course, $\delta = \tilde{\delta} \cup \{D \cap B\}$ is a division of $D$, for which we have

$$S_f(\delta) - s_f(\delta) = S_f(\tilde{\delta}) - s_f(\tilde{\delta}) + (M_B - m_B)\, \mu(D \cap B),$$

where

$$M_B = \sup \{f(x) : x \in D \cap B\} \quad \text{and} \quad m_B = \inf \{f(x) : x \in D \cap B\}.$$

Consequently $S_f(\delta) - s_f(\delta) < \dfrac{\varepsilon}{2} + 2M\dfrac{\varepsilon}{4M} = \varepsilon.$ ◇

Basicly the multiple integrals have the same properties as the simple ones (defined on compact sets from $\mathbb{R}$) ones:

**2.8. <u>Proposition.</u>** The integrable functions on *m.c.d.* have the properties:

(i) $\int\limits_D (\alpha f + \mu g) d\mu = \alpha \int\limits_D f d\mu + \beta \int\limits_D g d\mu$ (*linearity*)

(ii) $\int\limits_{D_1 \cup D_2} f d\mu = \int\limits_{D_1} f d\mu + \int\limits_{D_2} f d\mu$, whenever $\overset{\circ}{D}_1 \cap \overset{\circ}{D}_2 = \emptyset$ (*additivity relative to the domains*)

(iii) If $f \le g$ on $D$, then $\int\limits_D f d\mu \le \int\limits_D g d\mu$ (*monotony*).

(iv) If $f$ is integrable on $D$, then $|f|$ is also integrable on $D$, and:

$|\int\limits_D f d\mu| \le \int\limits_D |f| d\mu$ (*absolute integrability*)

(v) $\mu(D) \inf f(D) \le \int\limits_D f d\mu \le \mu(D) \sup f(D)$ (*mean-value property*)

The proof is directly based on definitions and it is omitted.

**2.9. <u>Proposition.</u>** If $f$ is continuous on the *m.c.d.* $D \subset \mathbb{R}^p$, $p \ge 1$, then there exists $\xi \in D$ such that $\int\limits_D f d\mu = f(\xi)\, \mu(D)$ (*mean-value integral formula*).

<u>Proof.</u> Because $f$ is continuous on the compact $D$, there exists $x_1, x_2 \in D$ such that $\inf f(D) = f(x_1)$ and $\sup f(D) = f(x_2)$. If we note

$$\lambda = \frac{1}{\mu(D)} \int_D f d\mu ,$$

then property (v) in proposition 2.8 takes the form $f(x_1) \le \lambda \le f(x_2)$.

Using the fact that $D$ is also open and connected, there exists a continuous curve $\gamma \subset D$ of end-points $x_1$ and $x_2$ . If $\varphi : [a, b] \to D$ is a parameterization of $\gamma$, then $g = f \circ \varphi : [a, b] \to \mathbb{R}$ is also continuous, hence it has the Darboux' property. In particular, because

$$g(a) = f(x_1) \le \lambda \le f(x_2) = =g(b),$$

it follows that there exists $t \in [a, b]$ such that $\lambda = g(t) = f(\xi)$, where $\xi = \varphi(t)$. Consequently, $\frac{1}{\mu(D)} \int_D f d\mu = f(\xi)$, for some $\xi \in D$. ◇

**2.10. Remark.** Mainly there are two methods for calculating the multiple integrals: one uses the reduction of the dimension by iteration; the other consists in changing the variables. We will analyze the first method starting out with the simplest case when $D \subset \mathbb{R}^{\mathbf{p}}$ reduces to a paralleloid $P$. More exactly, we consider a *Cartesian decomposition* of $P$ of the form $P = P' \times P''$, which leads to the distinction of two components in any $x \in P$, namely $x = (u, v)$, where $u = (x_1, ..., x_m) \in P'$ and $v = (x_{m+1}, ..., x_p) \in P''$ for some $1 \le m \le p - 1$. If $f : P \to \mathbb{R}$ , then we note $f(x) = f(u, v)$.

**2.11. Theorem.** Let $f : P \to \mathbb{R}$ be an integrable function on the paralleloid $P = P' \times P''$. If for each fixed $u \in P'$ there exists $I(u) = \int_{P''} f(u, v) dv$ , then $I : P' \to \mathbb{R}$ is an integrable function on $P'$, and the following equality holds:

$$\int_P f(x) dx = \int_{P'} I(u) du .$$

Proof. By dividing each side of $P'$ and $P''$ into $n$ equal parts, we obtain the divisions $\delta' = \{P'_1, ..., P'_{n^m}\}$ of $P'$, $\delta'' = \{P''_1, ..., P''_{n^{p-m}}\}$ of $P''$, and $\delta = \{P_{ij} = P'_i \times P''_j ; i = 1, ..., n^m; \ j = 1, ..., n^{p-m}\}$ of $P$. Let us note

$$m_{ij} = \inf f(P_{ij}) \quad \text{and} \quad M_{ij} = \sup f(P_{ij}),$$

so that for each $u_i \in P'_i$ and $v_j \in P''_j$ we have $m_{ij} \le f(u_i, v_j) \le M_{ij}$ .

Because $f$ is integrable relative to the variable $v$ on $P''$, it will be integrable relative to $v$ on $P''_j$ too, hence by integrating the above inequality we obtain

$$m_{ij}\, \mu''(P''_j) \le \int_{P''_j} f(u_i, v) dv \le M_{ij}\, \mu''(P''_j) ,$$

where $\mu''$ is Jordan's measure on $\mathbb{R}^{\mathbf{p-m}}$.

Adding these relations for all $j = 1, ..., n^{p-m}$, we obtain

$$\sum_{j=1}^{n^{p-m}} m_{ij}\,\mu''(P''_j) \le \int_{P''_j} f(u_i,v)dv = I(u_i) \le \sum_{j=1}^{n^{p-m}} M_{ij}\,\mu''(P''_j). \quad (*)$$

If μ' is Jordan's measure on $\mathbb{R}^{\mathbf{m}}$, it is easy to see that $\mu'(P'_i)\,\mu''(P''_j) = \mu(P_{ij})$ for all $i = 1, ..., n^m$ and $j = 1, ..., n^{p-m}$, where μ is the measure on $\mathbb{R}^{\mathbf{p}}$. Let us note by $\mathscr{S}' = \{u_1,...,u_{n^m}\}$ the system of intermediate points, and let

$$\sigma_I(\delta', \mathscr{S}') = \sum_{i=1}^{n^m} I(u_i)\,\mu'(P'_i)$$

be the Riemannian sums of $I$ on $P'$. By multiplying the relations (*) by $\mu'(P'_i)$, and adding all the forthcoming relations, we obtain

$$\sum_{i=1}^{n^m}\sum_{j=1}^{n^{p-m}} m_{ij}\,\mu(P_{ij}) \le \sigma_I(\delta', \mathscr{S}') \le \sum_{i=1}^{n^m}\sum_{j=1}^{n^{p-m}} M_{ij}\,\mu(P_{ij}) \quad (**)$$

Now we mention that $n \to \infty$ implies $\|\delta'\| \to 0$ and $\|\delta''\| \to 0$, as well as $\|\delta\| \to 0$ since $\|\delta\| \le \|\delta'\| + \|\delta''\|$. Consequently, because $f$ is integrable on $P$, the first and the last sums in the inequality (**) have a common limit, namely $\int_{P''} f^*(u,v)dv$, so it follows that the limit

$$\lim_{\|\delta'\|\to 0} \sigma_I(\delta', \mathscr{S}') = \int_{P'} I d\mu',$$

also exists, and $\int_P f d\mu = \int_{P'} I d\mu'$. ◇

**2.12. <u>Corollaries.</u>** (i) In the conditions of the above theorem we have:

$$\int_{P'xP''} f(u,v)dudv = \int_{P'}\left[\int_{P''} f(u,v)dv\right]du.$$

In particular, when $f(u, v) = g(u)h(v)$, we can reduce the integral of $f$ to a product of integrals of $g$ and $h$, i.e. $\int_P f d\mu = \left[\int_{P'} g(u)du\right]\left[\int_{P''} h(v)dv\right]$.

(ii) Interchanging $u$ and $v$ , if $J(v) = \int_{P'} f(u,v)du$ is integrable on $P'$, then

$$\int_P f d\mu = \int_{P''} J d\mu'',$$

or equivalently,

$$\int_{P'xP''} f(u,v)dudv = \int_{P''}\left[\int_{P'} f(u,v)dv\right]du.$$

(iii) If $m = p - 1$, i.e. $P'' = [a_p, b_p]$, we have

$$\int_P f d\mu = \int_{P'} \left[ \int_{a_p}^{b_p} f(u, x_p) dx_p \right] du ,$$

and by repeating the iteration, we obtain

$$\int_P f d\mu = \int_{a_1}^{b_1} \left[ \int_{a_2}^{b_2} \left[ \dots \int_{a_p}^{b_p} f(x_1, x_2, \dots, x_p) dx_p \dots \right] dx_2 \right] dx_1 .$$

All these formulas are direct consequences of the above theorem, so they need no proof. In particular, for double and triple integrals we have:

$$\iint_P f(x, y) dx dy = \int_{a_1}^{b_1} \left[ \int_{a_2}^{b_2} f(x, y) dy \right] dx ,$$

where $P = [a_1, b_1] \times [a_2, b_2]$, and respectively

$$\iiint_P f(x, y, z) dx dy dz = \int_{a_1}^{b_1} \left[ \int_{a_2}^{b_2} \left[ \int_{a_3}^{b_3} f(x, y, z) dz \right] dy \right] dx ,$$

where $P = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$.

**2.13. <u>Remark.</u>** The above formulas are rarely useful in practice because they refer to a very particular form of the domain *D*, namely that of a paralleloid. In order to extend these formulas up to an arbitrary *m.c.d.* $D \subset \mathbb{R}^p$ , $p > 1$, we introduce the notion of "section" as a generalization of the Cartesian decomposition of a paralleloid. More exactly, if $u = (x_1, ..., x_m)$ for some $m = 1, ..., p - 1$ is fixed, then the *n–section* of *D* is defined by

$$D[u] = \{v=(x_{m+1},...,x_p) : x = (u, v) \in D\}.$$

The set

$$Pr_m(D) = \{u=(x_1,...,x_m) : D[u] \neq \emptyset\}$$

represents the *m–projection* of *D*. Further we'll consider that the n–sections and the m–projections of *D* are also *m.c.* domains. In particular, when $m = p - 1$ we suppose that $D[u]$ reduces to a closed interval; more exactly, we say that *D* is *simple* iff there exist two functions $\varphi, \psi \in C_{\mathbb{R}}^{1}(Pr_{p-1}(D))$ such that $D[u] = [\varphi(u), \psi(u)]$ for all $u \in Pr_{p-1}(D)$.

**2.14. <u>Theorem.</u>** Let $D \subset \mathbb{R}^p$, $p > 1$, be a *m.c.d.* and let $f : D \to \mathbb{R}$ be integrable on *D*. If for each $u \in Pr_m(D)$ there exists $I(u)= \int_{D[u]} f(u, v) dv$, then $I : Pr_m(D) \to \mathbb{R}$ is integrable and $\int_D f d\mu = \int_{\mathrm{Pr}_m(D)} I(u) du$.

<u>Proof.</u> In order to reduce this theorem to theorem 2.12, let *P* be a paralleloid which contains *D*, and let $f^* : P \to \mathbb{R}$ be an extension of *f*, i.e.

$$f^*(x) = \begin{cases} f(x) & if\ x \in D \\ 0 & if\ x \in P \setminus D \end{cases}.$$

In this situation, $f^*$ is integrable on $P$, and $\int\limits_P f^* d\mu = \int\limits_D f d\mu$.

Because in the Cartesian decomposition $P = P' \times P''$ we have $P' = Pr_m(P)$ and $P' = P[u]$ for all $u \in P'$, theorem 2.12 takes the form

$$\int\limits_D f d\mu = \int\limits_{P'} I^*(u) du$$

where $I^*(u) = \int\limits_{P''} f^*(u,v) dv$. Now, it remains to see that

$$I^*(u) = \begin{cases} I(u) & \text{if } u \in \mathrm{Pr_m}(D) \\ 0 & \text{if } u \in P' \setminus \mathrm{Pr_m}(D) \end{cases},$$

and, because $f^*$ is null outside $D$, we have

$$I^*(u) = \int\limits_{D[u]} f(u,v) dv.$$

To conclude, we introduce this expression in the integral of $f$. ◇

In practice this theorem is mainly used for $m = 1$ and $m = p - 1$, when it furnishes the principal methods of iteration:

**2.15.** **<u>Method I of iteration.</u>** ($m = 1$) Let $[a_1, b_1] = Pr_1(D)$ be the projection on $x_1$–axis of $D$, and let us suppose that for any $x \in [a_1, b_1]$ there exists

$$I(x_1) = \int_{D[x_1]} f(x_1,..., x_p) dx_2 ... dx_p.$$

Then $I$ is integrable on $[a_1, b_1]$ and $\int\limits_D f d\mu = \int\limits_{a_1}^{b_1} I(x_1) dx_1$, i.e.

$$\int \underset{D}{\cdots} \int f(x_1,..., x_n) dx_1 ... dx_n = \int\limits_{a_1}^{b_1} \left[ \int \underset{D[x_1]}{\cdots} \int f(x_1, x_2,..., x_n) dx_2 ... dx_n \right] dx_1.$$

**2.16.** **<u>Method II of iteration.</u>** ($m = p - 1$) Let $D$ be a simple *m.c.d.* and let $f : D \to \mathbb{R}$ be an integrable function. If for any $u = (x_1,...,x_{p-1}) \in Pr_{p-1}(D)$ the function $x_p \mapsto f(u, x_p)$ is integrable on $[\varphi(u), \psi(u)]$, then the function $I : Pr_{p-1}(D) \to \mathbb{R}$, defined by

$$I(x_1,...,x_{p-1}) = \int\limits_{\varphi(x_1,...,x_{p-1})}^{\psi(x_1,...,x_{p-1})} f(x_1,..., x_{p-1}, x_p) dx_p,$$

is integrable on $Pr_{p-1}(D)$, and

$$\int_D f d\mu = \int_{\mathrm{Pr}_{p-1}(D)} I(x_1,..., x_{p-1}) dx_1 ... dx_{p-1},$$

i.e.

$$\int_D\!\!\dots\!\int f(x_1,\dots,x_p)dx_1\dots dx_p = \int_{\mathrm{Pr}_{p-1}(D)}\!\!\dots\!\int\left[\int_{\varphi(x_1,\dots,x_{p-1})}^{\psi(x_1,\dots,x_{p-1})} f(x_1,\dots,x_{p-1},x_p)dx_p\right]dx_1\dots dx_{p-1}$$

**2.17. Remarks.** (i) The above methods reduce the multiple integrals to simple integrals with variable limits. In the case of $p = 2$, when $D$ is simple *m.c.d.* in the plane, methods I and II coincide.

(ii) The above methods of iterating the multiple integrals can be intuitively described as techniques of "sweeping" the domain $D$ by different sections.

For example, if $D$ is a simple *m.c.d.* in the plane, then we may interpret the calculus of the integral $I(x) = \int_{\varphi(x)}^{\psi(x)} f(x,y)dy$ as finding a double integral on a thin band $B_x = \Delta x \times [\varphi(x), \psi(x)]$ from $D$ (see the figure VII.2.1).

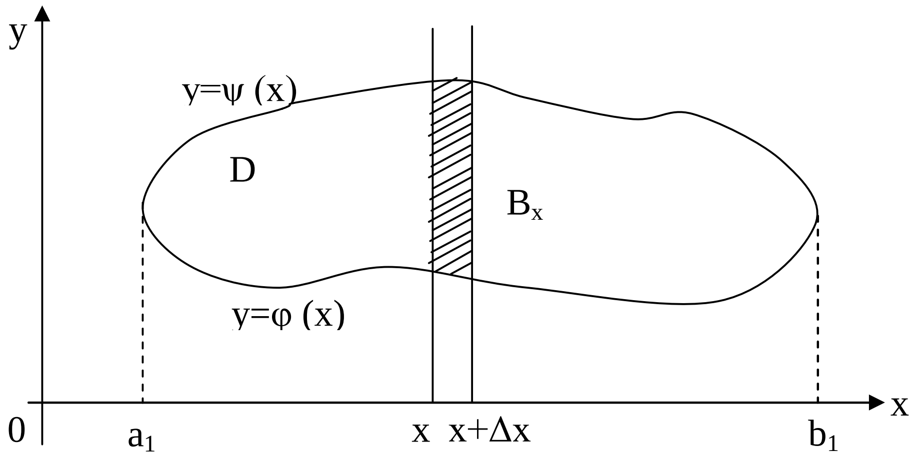



Fig. VII.2.1.

Finally, to obtain the double integral, the band $B_x$ "sweeps" the domain by a movement between $a_1$ and $b_1$, which means to calculate

$$\int_{a_1}^{b_1} I(x)dx = \iint_D f(x,y)dxdy\,.$$

Similarly, we can sweep the domain using horizontal bands, if $D$ allows.

(iii) Besides iteration, there is another technique of calculating multiple integrals, which is based on the change of variables. The formulas are similar to those concerning the simple integrals, but in the case of the multiple integrals we mainly use the change of the variables in order to transform the given domain $D$ into a simpler one, for example into a parallelepiped, if possible.

The following theorem of changing the variables in a multiple integral naturally extends the rule of changing the variable in the simple integral on $[a, b] \subset \mathbb{R}$. We recall that a *change of variable* of the open set $A \subseteq \mathbb{R}^P$ is a 1:1 diffeomorphism $T : A \to \mathbb{R}^P$ between $A$ and $B = T(A)$. As usually, we note $x = T(u) = (\varphi_1(u), ..., \varphi_p(u))$, where $u \in A$. According to the local inversion theorem, if the Jacobian of $T$ is non-null at $u_0$, then $T$ realizes a change of variables in a neighborhood of $u_0$ .

**2.18. <u>Theorem.</u>** Let $D, E \subset \mathbb{R}^P$, $p \in \mathbb{N}^*$ be *m.c.d.*, and let $T : E \to D$ be a transformation such that:

(i) $T$ is a 1:1

(ii) $T(E) = D$

(iii) $\det J_T(u) \neq 0$ at any $u \in \overset{\circ}{E}$

If $f : D \to \mathbb{R}$ is continuous on $D$, then

$$\int_D f(x)dx = \int_E (f \circ T)(u) \,|\det J_T(u)|\, du .$$

<u>Proof.</u> We may reason inductively relative to $p \in \mathbb{N}^*$. For $p = 1$ the property reduces to the well known theorem of changing the variable in the definite simple integral. Let us suppose that the theorem is valid up to $p = n - 1 \geq 1$. In order to prove it for $p = n$, we decompose the transformation $T : E \to D$ into $T = T_2 \circ T_1$ , where $T_1 : E \to \mathbb{R}^n$ is defined by

$$(v_1,...,v_n) = T_1(u_1, u_2,...,u_n) = (u_1, \varphi_2(u_1...,u_n), ..., \varphi_n(u_1,...,u_n))$$

and $T_2 : F \equiv T_1(E) \to \mathbb{R}^n$ is defined by

$$(x_1, ..., x_n) = T_2(v_1, v_2, ...,v_n) = (\varphi \circ T_1^{-1}(v_1,...,v_n), v_2, ..., v_n).$$

It is easy to see that $T_1$ and $T_2$ satisfy conditions (*i*)–(*iii*) if $T$ does, so the problem reduces to prove the assertion of the theorem for $T_1$ and $T_2$.

So we claim that

$$\int_D f(x_1,..., x_n)dx_1...dx_n = \int_F (f \circ T_2)(v_1,..., v_n) \,|\det J_{T_2}(v_1,..., v_n)|\, dv_1...dv_n \quad (*)$$

In fact, if $Pr_1(D) = [a_1 ,b_1]$ according to theorem 2.14 (method II),

$$\int_D f(x)dx = \int_{D[x_1]} \left[ \int_{a_1}^{b_1} f(x_1,..., x_n)dx_1 \right] dx_2...dx_n .$$

By changing the variable $x_1$ in the above simple integral (the case $p = 1$) we obtain

$$\int_{a_1}^{b_1} f(x_1, x_2,..., x_n)dx_1 = \int_{\alpha}^{\beta} f(\Phi_{(x_2,...x_n)}(v_1), x_2,..., x_n)\Phi'_{(x_2,...,x_n)}(v_1)dv_1$$

where $\Phi_{(x_2,...,x_n)}(v_1) = \varphi_1 \circ T_1^{-1}(v_1, x_2, ....,x_n)$, and $\Phi_{(x_2,...,x_n)}([\alpha, \beta]) = [a_1, b_1]$. Because

$$\Phi'_{(x_2,...,x_n)}(v_1) = \frac{\partial}{\partial x_1}(\varphi_1 \circ T_1^{-1})(v_1, x_2, ....,x_n) = det\, J_{T_2}(v_1, x_2, ....,x_n) \neq 0,$$

where $[\alpha, \beta] = Pr_1(F)$ and $D[x_1] = F[v_1]$ we obtain

$$\int_D f(x)dx = \int_{F[v_1]} \left[ \int_{\Pr_1(F)} (f \circ T_2)(v_1,...,v_n)\det J_{T_2}(v_1,...,v_n)dv_1 \right] dv_2...dv_n =$$

$$\int_F (f \circ T_2)(v_1,...,v_n)\,|\det J_{T_2}(v_1,...,v_n)|\,dv_1dv_2...dv_n$$

which proves (*) .

Now we note $g = (f \circ T_2)|\,det\, J_{T_2}|$, and we claim that

$$\int_F g(v)dv = \int_E (g \circ T_1)(u_1,...,u_n)\,|\det J_{T_1}(u_1,...,u_n)|\,du_1...du_n \,. \quad (**)$$

In fact, using again theorem 2.14 (method I), we can write

$$\int_F g(v)dv = \int_\alpha^\beta \left[ \int_{F[v_1]} g(v_1,...,v_n)dv_2...dv_n \right] dv_1,$$

where $F[v_1] \subset \mathbb{R}^{n-1}$. Because the property is supposed valid for $p = n - 1$, we obtain

$$\int_{F[v_1]} g(v_1,v_2,...,v_n)dv_2...dv_n =$$

$$\int_{E[u_1]} g(v_1,\varphi_2(v_1,u_2,...,u_n),...,\varphi_n(v_1,u_2,...,u_n))\,|\det \psi_{v_1}(u_2,...,u_n)|\,du_2...du_n$$

where $\psi_{v_1}(u_2, ..., u_n) = (\varphi_2(v_1, u_2, ...,u_n), ..., \varphi_n(v_1, u_2, ..., u_n))$. Consequently,

$$det\ J_{\psi_{v_1}}(u_2, ..., u_n) = det\ J_{T_1}(v_1, u_2, ...,u_n)\,.$$

In fact, $T_1$ preserves the first component which implies that $[\alpha, \beta] = Pr_1(E)$, hence

$$\int_F g(v)dv = \int_{\Pr_1(E)} \left[ \int_{E[u_1]} (g \circ T_1)(u_1,u_2,...,u_n)\,|\det J_{T_1}(u_1,u_2,...,u_n)|\,du_2...du_n \right] du_1 =$$

$$= \int_{\mathrm{E}} (\mathrm{g} \circ \mathrm{T}_1)(\mathrm{u})\,|\,\mathrm{det J}_{\mathrm{T}_1}(\mathrm{u})\,|\,du,$$

which is (**) .

Finally, combining (*) and (**) we obtain :

$$\int_D f(x)dx = \int_F g(v)dv = \int_E (g \circ T_1)(u)\,|\det J_{T_1}(u)|\,du =$$

$$= \int_E [(f \circ T_2) \circ T_1](u)\,|\det J_{T_2}(T_1u)|\,|\det J_{T_1}(u)|\,du =$$

$$= \int_E (f \circ T)(u) \,|\det J_T(u)|\, du,$$

which accomplishes the proof. ◇

**2.19. <u>Particular transformations.</u>** (i) The transitions to polar coordinates in the plane

$$\begin{cases} x = r\cos t \\ y = r\sin t \end{cases}$$

is a transformation $T : (0, \infty) \times [0, 2\pi) \to \mathbb{R}^2 \setminus \{(0, 0)\}$, for which *det* $J_T(r, t) = r$. Consequently, for any *m.c.d.* $D \subset \mathbb{R}^2 \setminus \{(0, 0)\}$ and continuous $f : D \to \mathbb{R}$ we have

$$\iint_D f(x, y)dxdy = \iint_{T^{-1}(D)} f(r\cos t, r\sin t) r dr dt.$$

(ii) Similarly, passing to the cylindrical coordinates in open space $(x,y,z) \mapsto (r,t, z)$,

$$\begin{cases} x = r\cos t \\ y = r\sin t \\ z = z \end{cases}$$

represents a transformation $T : (0, \infty) \times [0, 2\pi) \times \mathbb{R} \to \mathbb{R}^3 \setminus \{(0, 0,0)\}$ with

$$\det J_T(r, t, z) = r.$$

According to the previous theorem, for any *m.c.d.* $D \subset \mathbb{R}^3 \setminus \{(0, 0, 0)\}$ and any continuous $f : D \to \mathbb{R}$ we can write

$$\iiint_D f(x, y, z)dxdydz = \iiint_{T^{-1}(D)} f(r\cos t, r\sin t, z) r dr dt dz$$

(iii) The spherical coordinates in space are introduced by the formulas

$$\begin{cases} x = \rho\cos\varphi\sin\theta \\ y = \rho\sin\varphi\sin\theta \\ z = \rho\cos\theta \end{cases}$$

Considered as a transformation $T : (0, \infty) \times [0, 2\pi) \times [0, \pi] \to \mathbb{R}^3$, with

$$\det J_T(\rho, \varphi, \theta) = \rho^2 \sin\theta,$$

the change of variables $(x, y, z) \mapsto (\rho, \varphi, \theta)$ in the triple integral of a continuous function $f : D \to \mathbb{R}$ where $D \subset \mathbb{R}^3 \setminus \{(0, 0, 0)\}$ is a *m.c.d.*, is realized by the formula

$$\iiint_D f(x, y, z)dxdydz =$$

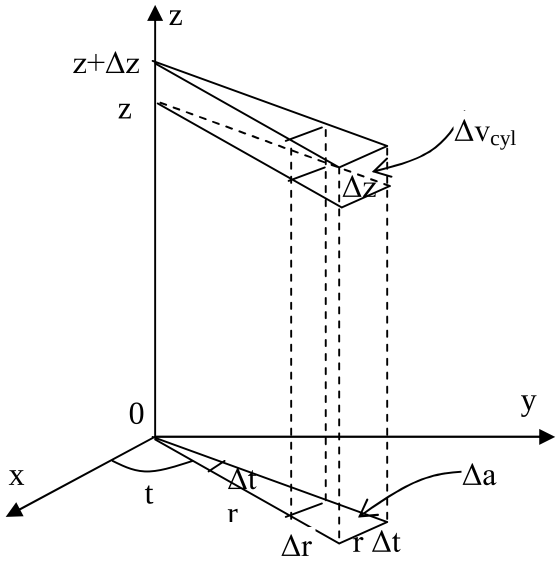
$$= \iiint_{T^{-1}(D)} f(\rho\cos\varphi\sin\theta, \rho\sin\varphi\sin\theta, \rho\cos\theta)\rho^2 \sin\theta d\rho d\varphi d\theta .$$

**2.20. <u>Remark.</u>** The change of the variables in a multiple integral formally reduces to the modification of the domain *D*, and to the replacement of the "differentials" according to the formula

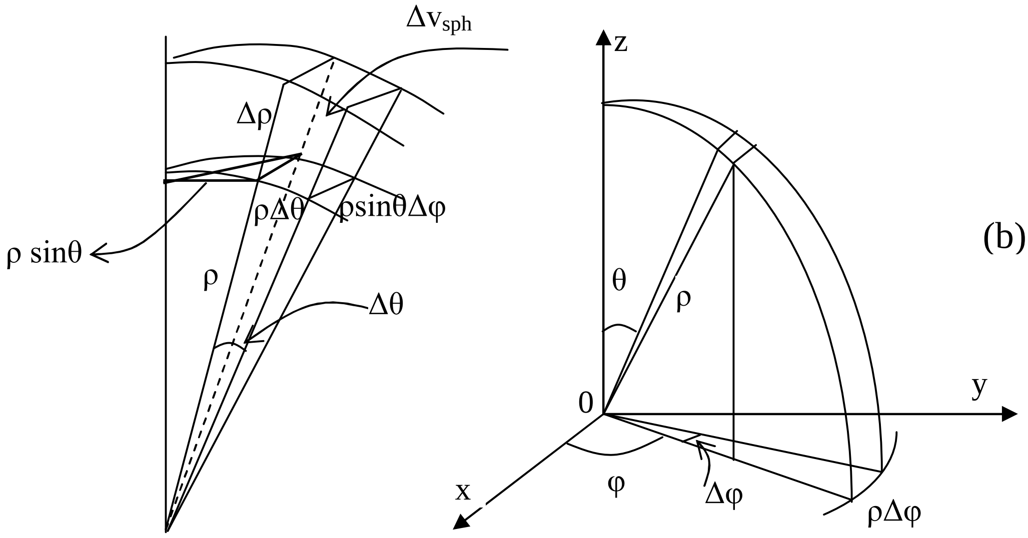
$$dx_1...dx_p = |\,det\ J_T(u_1, ...,u_p)|du_1....du_p.$$

This last equality may be considered a correspondence between the measures of the simplest elementary bodies in the considered coordinates. More exactly, in Cartesian coordinates $u_1$, ..., $u_p$, the paralleloid of sides $\Delta x_1$, ..., $\Delta x_p$, has the measure $\Delta\mu = |det\ J_T(u)|\ \Delta x_1, ..., \Delta x_p$. It is easy to see (Fig. VII.2.2) that in the above particular cases we have:

- $\Delta a = r\Delta r\Delta t$ for the area in polar coordinates in the plane;
- $\Delta v_{cyl} = r\Delta r\Delta t\Delta z$ for the volume in cylindrical coordinates in space;
- $\Delta v_{sph} = \rho^2 sin\theta\Delta\ \rho\Delta\ \varphi\Delta\theta$ for the volume in spherical coordinates .




(a)




(b)

Fig. VII.2.2

To close this paragraph we will analyze an important relation between double integrals and line integrals of the second order, which is known in the literature as *Green's formula*.

**2.21. Theorem.** Let $\gamma \subset \mathbb{R}^2$ be a simple, closed, piecewise smooth curve, which bounds the compact domain *D*, when it is traced once counter-clockwise. If $P, Q \in C^1_{\mathbb{R}}(\tilde{D})$, where $D \subset \tilde{D}$ ($\tilde{D}$ is open), and *D* has finite decompositions in simple sub-domains relative to the 0*x* as well as relative to the 0*y* axes, then the Green's formula holds:

$$\int_\gamma Pdx + Qdy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy .$$

Proof. It is clear that *D* can be decomposed into a finite number of rectangles and sub-domains of the form $D_1$, $D_2$, $D_3$ and $D_4$ as in the figure VII.2.3 (a) from below. Consequently it is sufficient to prove the formula for such simpler domains, e.g. for $D_1$.

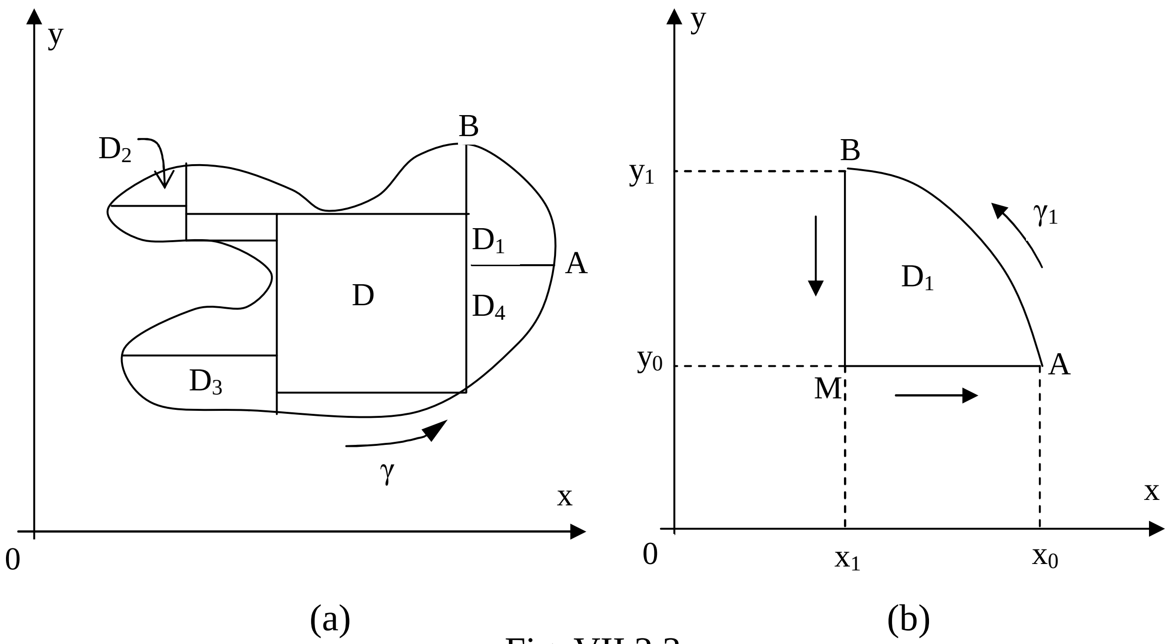



Fig. VII.2.3

In fact, using the two equations of $\gamma_1$, $y = \varphi(x)$, where $x \in [x_0, x_1]$, and $x = \psi(y)$, where $y \in [y_0, y_1]$, the double integral on $D_1$ becomes (see Fig. VII.2.3, (b))

$$\iint_{D_1} \frac{\partial Q}{\partial x} dxdy - \iint_{D_1} \frac{\partial P}{\partial y} dxdy = \int_{y_1}^{y_0} \left[ \int_{x_1}^{\psi(y)} \frac{\partial Q}{\partial x} dx \right] dy - \int_{x_1}^{x_0} \left[ \int_{y_0}^{\varphi(x)} \frac{\partial P}{\partial y} dy \right] dx =$$

$$= \int_{y_1}^{y_0} Q(\psi(y), y) dy - \int_{y_1}^{y_0} Q(x_1, y) dy - \int_{x_1}^{x_0} P(x, \varphi(x)) dx + \int_{x_1}^{x_0} P(x, y_0) dx =$$

$$= \int_{\gamma_1} Pdx + Qdy + \int_{\overline{BM}} Pdx + Qdy + \int_{\overline{MA}} Pdx + Qdy .$$

The other forms of the sub-domains are similarly discussed.

By adding such formulas, the line integrals on the interval segments cancel each other out, being calculated in opposite senses. ◇

**2.22. Corollary.** Under the conditions concerning $D$ in the above theorem, the area of $D$ has the expression

$$a(D) = \frac{1}{2}\int_{\gamma} xdy - ydx .$$

Proof. We can consider $P = -y$ and $Q = x$ in the above theorem, and take into consideration that $a(D) = \iint_D dxdy$. We recognize here the formula of Proposition 15, §3, Chapter VI, for more general shape of the domain. ◇

To conclude this section, we mention an interesting application of the double integrals in mechanics:

**2.23. Example.** A body $D$ of constant density $\gamma$ is obtained from a sphere of radius $R$ by removing a concentric sphere of radius $r < R$. We can show that the attraction of this solid on any material point lying in the interior sphere is null. In fact, using the spherical coordinates $(\rho, \varphi, \theta)$, the element of mass of $D$, say $\Delta M = \gamma\, \rho^2 \sin\theta \Delta\rho\, \Delta\varphi\, \Delta\theta$, acts on the mass $m$ with a force of value

$$\Delta F = k\frac{m\Delta M}{d^2},$$

where $d^2 = \rho^2 - 2r\,\rho\cos\theta + r^2$. The component along $oz$ is

$$\Delta F_z = \Delta F\, \cos(z,d) = \Delta F\, \frac{\rho\cos\theta - r}{d}.$$

Consequently

$$F_z = k\,\gamma m \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_1^{R} \frac{\rho^2 \sin\theta(\rho\cos\theta - r)}{(\rho^2 - 2r\rho\cos\theta + r^2)^{3/2}} d\rho =$$

$$= \frac{k\gamma m}{r} 2\pi \int_r^R \rho d\rho \int_0^{\pi} \frac{\rho r \sin\theta}{(\rho^2 - 2r\rho\cos\theta + r^2)^{3/2}} (\rho\cos\theta - r)d\theta ,$$

where the integral relative to $\theta$ can be computed by parts. Finally $F_z = 0$.

## PROBLEMS §VII.2

**1.** Depict the domains of integration and evaluate the following iterated integrals:

(i) $\int_0^1 dx \int_0^1 \frac{x^2 dy}{1+y^2}$ (ii) $\int_0^2 dy \int_0^1 (x^2+2y)dx$

(iii) $\int_1^2 dx \int_{1/2}^{x} \frac{x^2 dy}{y^2}$ (iv) $\int_0^1 dx \int_0^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\,dy$

Answers. (i) $\frac{\pi}{12}$; the integral breaks up into a product of simple integrals;

(ii) $\frac{14}{3}$; the domain is a rectangle, but the function differs from a product;

(iii) $\frac{9}{4}$; the function is a product $g(x)h(y)$, but $D$ is not a rectangle;

(iv) $\frac{\pi}{6}$; $D$ is a quarter of a disc and the integral is $\frac{1}{8}$ from the volume of the unit sphere.

**2.** Change the order of integration in the following double integrals:

(a) $\int_0^4 dx \int_{3x^2}^{12x} f(x,y)dy$ (b) $\int_0^1 dy \int_{-\sqrt{1-y^2}}^{1-y} f(x,y)dx$

Hint. (a) $\int_0^{48} dy \int_{y/12}^{\sqrt{y/8}} f(x,y)\,dx$; (b) Express the integral as a sum.

**3.** Evaluate $\iint\limits_D x\,dx\,dy$, where $D$ is:

(a) a triangle with vertices $O(0, 0)$, $A(1, 1)$, $B(0, 1)$;
(b) a region bounded by the straight line passing through the points $A(2, 0)$ and $B(0, 2)$, and by the arc of a circle of center $C(0, 1)$ and radius $r = 1$.
Hint. (a) $1/6$; (b) $1/6$.

**4.** Calculate $\iint\limits_D e^{\frac{x}{y}}\,dx\,dy$, where $D$ is a curvilinear triangle bounded by the curves of equations $y^2 = x$, $x = 0$, $y = 1$.
Answer. $1/2$.

**5.** Evaluate the integral $I = \iint\limits_D ydxdy$, where $D$ is bounded by the axis of abscissas and an arc of the cycloid $x = R(t - \sin t)$, $y = R(1 - \cos t)$.

Hint. $I = \int\limits_0^{2\pi R}\left[\int\limits_0^{\psi(x)} ydy\right]dx$, and we change the variable in the simple integral relative to $x$, i.e. $I = \int\limits_0^{2\pi R}\left[\int\limits_0^{R(1-\cos t)} ydy\right]R(1-\cos t)dt = \frac{5\pi}{2}R^3$.

**6.** Calculate $I = \iiint\limits_D f(x,y,z)dxdydz$ if $D = [0, 1]^3$ is the unit cube and:

(i) $f(x, y, z) = xy^2e^z$  (ii) $f(x, y, z) = \dfrac{1}{\sqrt{x+y+z+1}}$.

Hint. (i) $I$ is a product of simple integrals. (ii) Use theorem 2.11.

**7.** Calculate $I = \iiint\limits_D xyzdxdydz$ if:

(i) $D$ is a tetrahedron bounded by the planes $x + y + z=1$, $x = 0$, $y = 0$ and $z = 0$.

(ii) $D$ is a region between the cone $z = \sqrt{x^2+y^2}$ and the paraboloid
$z = 1 - x^2 - y^2$.

Hint. (i) $I = \int\limits_0^1 x\left[\int\limits_0^{1-x} y\left[\int\limits_0^{1-x-y} zdz\right]dy\right]dx = \frac{1}{720}$; (ii) $\iint\limits_{\mathrm{Pr}_2(D)} xy\left[\int\limits_{\sqrt{x^2+y^2}}^{1-x^2-y^2} zdz\right]dxdy$,

where $Pr_2(D) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le r^2\}$, and $r = \frac{\sqrt{5}-1}{2}$. Alternatively, pass to cylindrical (or polar) coordinates.

**8.** Evaluate $I = \iiint\limits_D zdxdydz$, where $D$ is bounded by the plane $z = 0$ and:

(i) the upper half of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

(ii) the pyramid $|x| + |y| + |z| = 1$, $z \ge 0$.

Hint. Use the formula of Method I, namely $I = \int\limits_{z_0}^{z_1} z\left[\iint\limits_{D[z]} dxdy\right]dz$, where the double integral represents the area of a simple section.

**9.** Evaluate:

(i) $\int_0^2 dx \int_0^{\sqrt{2x-x^2}} dy \int_0^a z\sqrt{x^2+y^2}\,dz$ using cylindrical coordinates

(ii) $\int_{-R}^{+R} dx \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy \int_0^{\sqrt{R^2-x^2-y^2}} (x^2+y^2)dz$ using spherical coordinates.

Answers. (i) $\frac{8}{9}a^2$ ; (ii) $\frac{4}{15}\pi R^5$.

**10.** Passing to polar coordinates, evaluate $I = \iint_D \sqrt{a^2 - x^2 - y^2}\,dxdy$, where $D$ is a loop of the lemniscate $(x^2 + y^2)^2 = a^2(x^2 - y^2)$, $x \geq 0$.
Hint. Draw the corresponding domain bounded by $r^2 = a^2 cos2t$ in the plane $(r, t)$, $t \in [0, \pi]$.

**11.** Calculate $\iint_D \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\,dxdy$, extended over the region $D$, which is bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
Hint. Use the generalized polar coordinates $(r, t)$, defined by

$$\begin{cases} \frac{x}{a} = r\cos t \\ \frac{y}{b} = r\sin t \end{cases}.$$

**12.** Evaluate $I = \iint_{[0,1]x[0,1]} (x^2 - y^2)e^{4xy}dxdy$ using the coordinates $u = x + y$ and $v = x - y$.
Hint. Divide the square in the $(u, v)$–plane into two triangles.

**13.** Show that there is an infinite area between any two hyperbolas $x^2 - y^2 = r_1^2$ and $x^2 - y^2 = r^2_2$, $x > 0$, $r_1 > 0$, $r_2 > 0$.
Hint. Use the change $x = r \cosh t$, $y = r \sinh t$ .

**14.** Identify the domains and evaluate their areas:

(i) $\int_{-1}^{2} dx \int_{x^2}^{x+2} dy$  (iii) $\int_{\frac{\pi}{4}}^{arctg 2} dt \int_{0}^{a \sec t} r dr$

(ii) $\int_{0}^{2} dy \int_{a-y}^{\sqrt{a^2-y^2}} dx$  (iv) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dt \int_{a}^{a(1+\cos t)} r dr$

Answers. (i) $\frac{9}{2}$ ;(*ii*) $\frac{a^2}{4}(\pi-2)$; (iii) $\frac{9}{2}$; (iv) $\frac{a^2}{4}(\pi+8)$.

**15.** Find the volume of the body bounded by the *xy*–plane, the sphere $x^2 + y^2 + z^2 = a^2$ and the cylinder $x^2 + y^2 = ax$ .
Hint. Independently of the use of a double or a triple integral, the volume is expressed by the integral $\iint_D \sqrt{a^2 - x^2 - y^2}\, dxdy$, where *D* is the interior of the disc $x^2 + y^2 = ax$. Passing to polar coordinates, when *D* is bounded by $r = a \cos t$, $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, it reduces to

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\int_{0}^{a\cos t} \sqrt{a^2 - r^2}\, r dr\right] dt = \frac{a^3}{9}(3\pi - 4).$$

**16.** Using the Green's formula, evaluate

$$I = \int_{\gamma} \sqrt{x^2 + y^2}\, dx + y\left[xy + \ln(x + \sqrt{x^2 + y^2})\right] dy,$$

where $\gamma$ consists of the graphs of $y = \cos x$ and $y = \sin x$ for $x$ between $\frac{\pi}{4}$ and $5\pi/4$.

Hint. $I = \iint_D y^2 dxdy = \int_{\pi/4}^{5\pi/4} \left[\int_{\cos x}^{\sin x} y^2 dy\right] dx.$

**17.** Evaluate the line integral $I = \int_{\gamma} \frac{xdy - ydx}{x^2 + y^2}$, where $\gamma$ is a circle traced *n* times counter-clockwise, and:
a) the origin is lying outside $\gamma$
b) the origin is lying inside $\gamma$ .

Hint. a) According to Green's formula, $I = 0$. b) The Green's formula is not valid any more, but a direct calculation of the line integral gives $I = 2\pi$.

**18.** Find the mass and the center of gravity of the solid body bounded by the paraboloid $y^2 + 2z^2 = 4x$ and the plane $x = 3$, whose density is

$$\rho(x, y, z) = \left(1+x^2\right)^{-1}.$$

Hint. The mass is $M = \iiint\limits_D \rho dxdydz = \int\limits_0^3 \frac{1}{1+x^2}\left[\iint_{D[x]} dydz\right]dx$, where $D[x]$ is an elliptic lamina of semi-axes $2\sqrt{x}$ and $\sqrt{2x}$, hence the double integral is $a(D[x]) = 2\pi x\sqrt{2}$. The coordinates of the center of gravity are

$$x_G = \frac{1}{M}\iiint\limits_D x\rho(x,y,z)dxdydz,$$

and $y_G = z_G = 0$ (because of symmetry).

**19.** A solid circular cone has the radius of the base equal to $R$, the altitude $h$, and a constant density $\rho$. Find its moment of inertia relative to a diameter of the base.

Hint. Take the plane of the base as $xoy$ and the axis of symmetry as $oz$. Evaluate $I_x = \iiint_D (y^2+z^2)\rho dxdydz$ using cylindrical coordinates. The result is $I_x = \frac{\pi\rho h R^2}{60}(2h^2+3R^2)$.

**20.** Show that the force of attraction exerted by a homogeneous sphere on an external material point does not change if the entire mass of the sphere is concentrated at its center.

Hint. Let $M$ (= $\frac{4}{3}\pi R^3 \gamma$) be the mass of the sphere of density $\gamma$ and radius $R$. Putting the origin of the coordinates in the center of the sphere, and the mass $m$ on the $oz$–axis, at the distance $L$ to the origin, in cylindrical coordinates, the distance between $m$ and the current point $(r, t, z)$ of the sphere ($r \le R$) will be $d = \sqrt{r^2+(L-z)^2}$. The elementary force has the value $\Delta F = k\frac{m\gamma\Delta v}{d^2}$, where $\Delta v = r\,\Delta r\,\Delta t\,\Delta z$. Because of symmetry, we are interested in finding the $z$–component of this force

$$\Delta F_z = \Delta F\, cos(d,z) = \Delta F\,\frac{L-z}{d}.$$

Evaluating the triple integral, it follows that $F = kmM/L^2$.

## § VII.3. IMPROPER MULTIPLE INTEGRALS

Up to now we have considered multiple integrals of bounded functions on compact domains in $\mathbb{R}^p$. These integrals correspond to the definite integral on $\mathbb{R}$, and they are called *integrals on compact domains*. As in the case of a simple integral on an interval of $\mathbb{R}$, there are situations when we must evaluate multiple integrals of non-bounded functions, or on non-bounded sets. All these situations are included in the study of integrability on non-compact sets.

**3.1. <u>Definition.</u>** Let $\Omega \subseteq \mathbb{R}^p$ be a non-compact domain for which each bounded part of the frontier is negligible. We say that a sequence $(D_n)_{n\in\mathbb{N}}$ of measurable compact domains (briefly *m.c.d.*) is *exhausting* $\Omega$ iff for any compact $K \subset \Omega$ there exists $n_0 \in \mathbb{N}$ such that $K \subset D_n$ for all $n \geq n_0$ .

As for arbitrary sequences of sets, we say that $(D_n)_{n\in\mathbb{N}}$ is *increasing* iff $D_n \subseteq D_{n+1}$ for all $n \in \mathbb{N}$.

**3.2. <u>Examples.</u>** (i) The domain $\Omega = \mathbb{R}^3$ is exhausted by each of the sequences $(D_n)_{n\in\mathbb{N}}$ , $(E_n)_{n\in\mathbb{N}}$ and $(F_n)_{n\in\mathbb{N}}$ of *m.c.d.*, where

$D_n = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq n^2\}$

$E_n = \{(x, y, z) \in \mathbb{R}^3 : |x| + |y| + |z| \leq n\}$

$F_n = \{(x, y, z) \in \mathbb{R}^3 : max\{|x|, |y|, |z|\} \leq n\}$

(ii) The sequence of *m.c.d.* $(D_n)_{n\in\mathbb{N}}$ of the form

$D_n = \{(x, y) \in\mathbb{R}^2 : \frac{1}{n^2} \leq x^2 + y^2 \leq 1\}$ is exhausting the non-compact domain $\Omega = S(0, 1) \setminus \{(0, 0)\}$, which is the unit disk without center.

(iii) In the plane $(\rho, \theta)$, the infinite band $\Omega = [0, \infty) \times [0, 2\pi]$ is exhausted by the sequence of *m.c.* domains of the form

$$D_n = \{(\rho, \theta) \in\mathbb{R}^2 : 0\leq \rho \leq n, 0 \leq \theta\leq 2\pi\}.$$

**3.3. <u>Definition.</u>** Let $\Omega \subseteq \mathbb{R}^p$ be a non-compact domain, and let $f : \Omega \to \mathbb{R}$ be integrable on each *m.c.d.* $D \subset \Omega$. We say that $f$ is *improperly integrable* on $\Omega$ iff for every increasing sequence of *m.c.d.*, $(D_n)_{n\in\mathbb{N}}$, which is exhausting $\Omega$, the sequence $\left(\int_{D_n} f d\mu\right)_{n\in\mathbb{N}}$ is convergent (see later that its limit does not depend on the particularly chosen sequence $(D_n)_{n\in\mathbb{N}}$). In such a case we note

$$\lim_{n\to\infty} \int_{D_n} f d\mu = \int_{\Omega} f d\mu$$

and we call it *improper integral of f on D*. Alternatively we say that the integral of $f$ on $\Omega$ is *convergent*.

The correctness of the above definition is based on the following property:

**3.4. <u>Proposition.</u>** If $f$ is (improperly) integrable on $\Omega$, then $\int\limits_{\Omega} f d\mu$ does not depend on the particular increasing and exhausting sequence of *m.c.* domains $(D_n)_{n\in\mathbb{N}}$, for which we calculate the limit of numerical sequence

$$\left(\int\limits_{D_n} f d\mu\right)_{n\in\mathbb{N}}.$$

<u>Proof.</u> Let $(D_n)_{n\in\mathbb{N}}$ and $(E_n)_{n\in\mathbb{N}}$ be two increasing sequences of *m.c.* domains which exhaust $\Omega$. By hypothesis, $I = \lim\limits_{n\to\infty}\int\limits_{D_n} f d\mu$ and $J = \lim\limits_{n\to\infty}\int\limits_{E_n} f d\mu$ exist. The problem is to show that $I = J$.

In fact, because both $(D_n)_{n\in\mathbb{N}}$ and $(E_n)_{n\in\mathbb{N}}$ are increasing and exhausting, for each $n \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that $D_n \subseteq E_k$. Similarly, for $k \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $E_k \subseteq D_m$, and for $m \in \mathbb{N}$ there exists $l \in \mathbb{N}$ such that $D_m \subseteq E_l$, etc. On this way we obtain an increasing and exhausting sequence of *m.c.* domains

$$D_1 \subseteq \ldots \subseteq D_n \subseteq E_k \subseteq D_m \subseteq E_l \subseteq \ldots$$

for which, according to the hypothesis, the sequence of integrals

$$\int\limits_{D_1} f d\mu, \ldots, \int\limits_{D_n} f d\mu, \int\limits_{E_k} f d\mu, \int\limits_{D_m} f d\mu, \int\limits_{E_l} f d\mu, \ldots.$$

is convergent. Because this convergent sequence contains subsequences of the convergent sequences

$$\left(\int\limits_{D_n} f d\mu\right)_{n\in\mathbb{N}} \text{ and } \left(\int\limits_{E_k} f d\mu\right)_{n\in\mathbb{N}},$$

it follows that all these sequences have the same limit, hence in particular we obtain the designed equality $I = J$. ◊

**3.5. <u>Remarks.</u>** (i) Because $\mathbb{R}$ is a complete metric space, the sequence $\left(\int\limits_{D_n} f d\mu\right)_{n\in\mathbb{N}}$ is convergent if and only if it is fundamental. In addition, because $(D_n)_{n\in\mathbb{N}}$ is an increasing sequence, and the multiple integral is additive relative to the domains, the above theorem may be formulated as follows: The integral $\int\limits_{\Omega} f d\mu$ is convergent if and only if there exists an increasing and exhaustive sequence $(D_n)_{n\in\mathbb{N}}$ of *m.c.d.*, such that for any $\varepsilon > 0$ there exists $n(\varepsilon) \in \mathbb{N}$ such that for any $n > n(\varepsilon)$ and $m \in \mathbb{N}$ we have $\left| \int_{D_{n+m}\setminus D_n} f d\mu \right| < \varepsilon$.

(ii) Taking as model the simple improper integrals, the study of the multiple improper integrals can be done in terms of numerical series with elements of the form $\int_{D_{n+1}\setminus D_n} f d\mu$ . The general properties of the multiple integral remain valid for improper integrals:

**3.6. <u>Proposition.</u>** (i) If $f, g : \Omega \to \mathbb{R}$ are improperly integrable on $\Omega$ and $\alpha, \beta \in \mathbb{R}$ , then $\alpha f + \beta g$ is also integrable on $\Omega$ and

$$\int_\Omega (\alpha f + \beta g) d\mu = \alpha \int_\Omega f d\mu + \beta \int_\Omega g d\mu \ \ (\textit{linearity})$$

(ii) Let $\Omega_1$, $\Omega_2$ and $\Omega = \Omega_1 \cup \Omega_2$ be non-compact domains for which $\overset{\circ}{\Omega}_1 \cap \overset{\circ}{\Omega}_2 = \emptyset$. If $f : \Omega \to \mathbb{R}$ is improperly integrable on $\Omega_1$ and $\Omega_2$ , then it is integrable on $\Omega$ and

$$\int_\Omega f d\mu = \int_{\Omega_1} f d\mu + \int_{\Omega_2} f d\mu \ \ \text{(additivity relative to the domains).}$$

<u>Proof.</u> (i) The same relation holds on any compact $K \subset \Omega$.

(ii) If $(D_n)_{n\in\mathbb{N}}$ and $(E_n)_{n\in\mathbb{N}}$ are increasing and exhausting sequences of *m.c.d.*s for $\Omega_1$ and $\Omega_2$ , then $(D_n \cup E_n)_{n\in\mathbb{N}}$ is increasing and exhausting for $\Omega$, and

$$\int_{D_n \cup E_n} f d\mu = \int_{D_n} f d\mu + \int_{E_n} f d\mu$$

holds for all $n \in \mathbb{N}$. ◇

In particular, the convergence of improper integrals of a positive function can be easily studied:

**3.7. <u>Theorem.</u>** (*Boundedness criterion of convergence*) The positive function $f : \Omega \to \mathbb{R}^+$ is improperly integrable if and only if there exists an increasing and exhausting sequence $(D_n)_{n\in\mathbb{N}}$ of *m.c.d.*s for which the sequence $\left(\int_{D_n} f d\mu\right)_{n\in\mathbb{N}}$ is bounded.

<u>Proof.</u> Because $f$ is positive and $(D_n)_{n\in\mathbb{N}}$ is increasing, it follows that the sequence $\left(\int_{D_n} f d\mu\right)_{n\in\mathbb{N}}$ is increasing too, hence it is convergent if and only if it is bounded. ◇

**3.8. <u>Proposition.</u>** (i) Let the functions $f, g : \Omega \to \mathbb{R}^+$ satisfy $f \le g$. If $f$ and $g$ are integrable on $\Omega$, then $\int_\Omega f d\mu \le \int_\Omega g d\mu$.

(ii) If $f : \Omega \to \mathbb{R}^+$ is improperly integrable on $\Omega$, and on some subset $\Omega' \subseteq \Omega$, then the following inequality holds

$$\int_{\Omega'} f d\mu \le \int_\Omega f d\mu.$$

<u>Proof.</u> (i) For every measurable compact domain $K \subset \Omega$, we have

$$\int_K f d\mu \le \int_K g d\mu.$$

(ii) Let us consider $h : \Omega \to \mathbb{R}$, of values,

$$h(x) = \begin{cases} 1 & \text{if } x \in \Omega' \\ 0, & \text{if } x \in \Omega \setminus \Omega'. \end{cases}$$

Of course $0 \le hf \le f$ and $h f = f$ on $\Omega'$. Because $f$ is integrable on $\Omega$, and $hf$ is integrable on $\Omega'$, according to (i) we obtain $\int_\Omega hf d\mu \le \int_\Omega f d\mu$.

It remains to see that $\int_\Omega hf d\mu = \int_{\Omega'} f d\mu$. ◇

**3.9. <u>Remark.</u>** In the case of a simple improper integral on domains $I \subseteq \mathbb{R}$, we have seen that $\int_I f(x) dx$ may be convergent without $\int_I |f(x)| dx$, so it makes sense to discuss about semi-convergence, and absolute convergence. This property has no analogue in the theory of multiple integrals. In fact, according to the following theorem, the integrals $\int_D f d\mu$ and $\int_D |f| d\mu$ are simultaneously convergent (respectively divergent). Consequently, it is a nonsense to speak of semi-convergent improper multiple integrals. It is not wrong to speak of *absolutely convergent* integrals, but this notion coincides with that of simple convergence.

In order forms to study the relation between "convergence" and "absolute convergence" for multiple integrals for arbitrary $f : \Omega \to \mathbb{R}$, we will define the *positive* and the *negative* part of $f$ by

$$f_+ = \frac{1}{2}[|f| + f]; \quad f_- = \frac{1}{2}[|f| - f].$$

It is clear that both $f_+$ and $f_-$ are positive, but smaller that $|f|$. In addition, we obviously have $f = f_+ - f_-$, and $|f| = f_+ + f_-$.

**3.10. <u>Theorem.</u>** Let us consider that $f : \Omega \to \mathbb{R}$ is integrable on any *m.c.d.* $D \subset \Omega$. Then $f$ is improperly integrable on $\Omega$ if and only if $|f|$ is.

Proof. At the very beginning we mention that $f$ is properly integrable on any *m.c.d.* $D \subset \Omega$ iff $|f|$ is , so the statement of the theorem essentially refers to the improper integrability on the non-compact domain $\Omega$. In fact, for arbitrary $x', x'' \in D$ we have

$$\big|\, |f(x'')| - |f(x')| \,\big| \leq |f(x'') - f(x')|,$$

hence $|f|$ has a smaller oscillation than $f$ on any division of $D$. It remains to use the Darboux criterion of integrability.

Let us suppose that $|f|$ is integrable. Because $f_+, f_- \leq |f|$, according to proposition 38, the integrability of $|f|$ implies that of $f_+$ and $f_-$. Using the property of linearity, it follows that $f$ is also integrable.

Conversely, let us suppose that $f$ is improperly integrable on $\Omega$, but $|f|$ is not integrable. Since $|f| \geq 0$, this means that for any sequence $(D_n)_{n\in\mathbb{N}}$ of monotonically exhausting *m.c.d.*s in $\Omega$, we have $\lim\limits_{n\to\infty} \int\limits_{D_n} |f|\, d\mu = +\infty$.

By rearranging the convenient indices, if necessary, we can consider that the successive terms $D_n$ and $D_{n+1}$ are chosen so that

$$\int\limits_{D_{n+1}} |f|\, d\mu > 3 \int\limits_{D_n} |f|\, d\mu + 2n$$

for any $n \in \mathbb{N}$. Denoting $A_n = \overline{D_{n+1} \setminus D_n}$, and using the additivity of the multiple integral, this inequality becomes $\int\limits_{A_n} |f|\, d\mu > 2 \int\limits_{D_n} |f|\, d\mu + 2n$ for all $n \in \mathbb{N}$. Because $f$ (and also $|f|$) is properly integrable on $D_n$, it follows that $f_+$ and $f_-$ are also properly integrable, and since $|f| = f_+ + f_-$, we obtain

$$\int\limits_{A_n} |f|\, d\mu = \int\limits_{A_n} f_+ d\mu + \int\limits_{A_n} f_- d\mu.$$

Now let us suppose that

$$\int\limits_{A_n} f_+ d\mu \geq \int\limits_{A_n} f_- d\mu. \qquad (*)$$

In this case

$$2 \int\limits_{A_n} f_+ d\mu \geq \int\limits_{A_n} |f|\, d\mu$$

hence, according to the previous inequality,

$$\int\limits_{A_n} f_+ d\mu > \int\limits_{D_n} |f|\, d\mu + n.$$

Now, let $B_n$ be a closed part of $A_n$ on which $f_+ = f$, such that $\int\limits_{A_n} f_+ d\mu = \int\limits_{B_n} f d\mu$. Then $\int\limits_{B_n} f d\mu > \int\limits_{D_n} |f|\, d\mu + n$.

Adding this inequality to the obvious one $\int\limits_{D_n} f d\mu > -\int\limits_{D_n} | f \,| d\mu$, we obtain

$$\int\limits_{D_n \cup B_n} f d\mu > n .$$

Similarly, if instead of (*) we admit its contrary, we would obtain that

$$\int\limits_{D_n \cup B_n} f d\mu < -n.$$

Finally, it remains to see that $(E_n)_{n\in\mathbb{N}}$, where $E_n = D_n \cup B_n$ is an increasing and exhausting sequence of *m.c.d.*, for which $|\int\limits_{E_n} f d\mu| > n$, hence the sequence $(\int\limits_{E_n} f d\mu)_{n\in\mathbb{N}}$ cannot be convergent. ◇

**3.11. <u>Remarks.</u>** (i) Because the study of the improper integrability of a positive function (like $| f |$) is easier, the above theorem simplifies the problem of convergence for the integral of functions which do not maintain the sign.

(ii)The convergence of a multiple integral is sometimes considered in the sense of the *principal value*. This means that the increasing and exhausting sequence of *m.c.d.* $(D_n)_{n\in\mathbb{N}}$ consists of "spherical sets". More exactly:

a) when $\Omega = \mathbb{R}^p$, we take $D_n = \{x\in \mathbb{R}^p : ||x|| \le n\}$, and

b) when $\Omega = K \setminus \{x_0\}$, when $K$ is a compact domain for which $x_0 \in \overset{\circ}{K}$, then

$$D_n = K \setminus \{ x\in \mathbb{R}^p : ||x - x_0|| < \frac{r}{n} \},$$

where $r$ is chosen in order to have $S(x_0 , r) \subset \overset{\circ}{K}$.

(i) Before calculating an improper multiple integral it is necessary to check the convergence of the respective integral, since a particular way of carrying out the calculation may lead to a convergent process, even though the integral is divergent. Therefore it is advisable to use the methods of calculating multiple integrals (iteration, change of variables, etc.) just on compact domains, but not on the whole non-compact domain. In other terms, the simple integrals, which occur when using some method of evaluating a multiple integral, might be convergent even for non-convergent multiple integrals.

## PROBLEMS § VII.3

**1.** (i) Show that $I = \iint\limits_{\mathbb{R}^2} e^{-(x^2+y^2)}dxdy$ is convergent;

(ii) Evaluate $I$ using polar coordinates;

(iii)Deduce the value of $J = \int\limits_{-\infty}^{+\infty} e^{-x^2}dx$.

Hint. Function $f : \mathbb{R}^2 \to \mathbb{R}$, expressed by $f(x, y) = e^{-x^2-y^2}$ is positive, hence it is sufficient to show that all the integrals

$$I_n = \iint\limits_{D_n} e^{-x^2-y^2}dxdy,$$

are bounded, where $D_n = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le n^2\}$, $n \in \mathbb{N}$. In fact, using polar coordinates $(r, t)$ we obtain

$$I_n = 2\pi\int\limits_0^n e^{-r^2}rdr = \pi(1-e^{-n^2}) < \pi.$$

(ii) $I = \lim\limits_{n\to\infty} I_n = \pi$.

(iii) Iterating in Cartesian coordinates we obtain $J^2 = I$, hence $J = \sqrt{\pi}$.

**2.** Show that function $f : \mathbb{R}^2 \to \mathbb{R}$, of values $f(x, y) = sin\,(x^2 + y^2)$ is not improperly integrable on $\mathbb{R}^2$.

Hint. If we note

$$D_n = \{(\mathrm{x, y}) \in \mathbb{R}^2 : x^2 + y^2 \le 2n\,\pi\}$$

and

$$E_n = \{(x, y) \in \mathbb{R}^2 : x^2+y^2 \le 2n\,\pi + \frac{\pi}{2}\},$$

then, for any $n \in \mathbb{N}$, we have $\int\limits_{D_n} fd\mu = 0$, while $\int\limits_{E_n} fd\mu = \pi$.

**3.** Study the convergence of the integrals

$$I(\alpha) = \iint\limits_{\Omega_2} \frac{dxdy}{(x^2+y^2)^\alpha}, \text{ and } J(\alpha) = \iiint\limits_{\Omega_3} \frac{dxdydz}{(x^2+y^2+z^2)^\alpha},$$

where

$$\Omega_2 = \{(\mathrm{x, y}) \in \mathbb{R}^2 : x^2 + y^2 \le 1\},$$

and

$$\Omega_3 = \{(\mathrm{x\ y, z}) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \ge 1\},\ \alpha \in \mathbb{R}.$$

Hint. Both $I(\alpha)$ and $J(\alpha)$ refer to positive functions, hence we can apply theorem 3.7. Denoting $D_n = \{(x, y) \in \Omega_2 \ : 1 \le x^2 + y^2 \le n^2\}$, $n \in \mathbb{N}^*$, and passing to polar coordinates, we obtain

$$I_n(\alpha) = \iint_{D_n} \frac{dxdy}{(x^2+y^2)^\alpha} = 2\pi \int_1^n r^{1-2\alpha} dr = \frac{\pi}{1-\alpha}[n^{2(1-\alpha)} - 1],$$

hence $I(\alpha)$ is convergent for $\alpha > 1$, and divergent for $\alpha \le 1$.

Similarly, considering $E_n = \{(x\ y, z) \in \Omega_3 : 1 \le x^2 + y^2 + z^2 \ge n^2\}$, where $n \in \mathbb{N}^*$, and using spherical coordinates, we obtain

$$J_n(\alpha) = \iiint_{E_n} \frac{dxdydz}{(x^2+y^2+z^2)^\alpha} = 4\pi \int_1^n \rho^{1-2\alpha} d\rho = \frac{4\pi}{3-2\alpha}[n^{3-2\alpha} - 1],$$

hence $J(\alpha)$ is convergent for $\alpha > \frac{3}{2}$, and divergent if $\alpha \le \frac{3}{2}$. The cases $\alpha = 1$ in $I_n$, and $\alpha = \frac{3}{2}$ in $J_n$, must be separately discussed.

**4.** Study the convergence of the integrals

$$I(\beta) = \iint_{\Sigma_2} \frac{dxdy}{(x^2+y^2)^\beta}, \text{ and } J(\beta) = \iiint_{\Sigma_3} \frac{dxdydz}{(x^2+y^2+z^2)^\beta},$$

where

$$\Sigma_2 = \{(x, y) \in \mathbb{R}^2 : 0 < x^2 + y^2 \le 1\},$$
$$\Sigma_3 = \{(x\ y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \le 1\}$$

and $\beta$ is a real parameter.

Hint. On any compact $K_n = \{(x, y) \in \Sigma_2 : x^2 + y^2 \ge \frac{1}{n^2}\}$, $n \in \mathbb{N}^*$, using polar coordinates, we have

$$I_n(\beta) = \iint_{K_n} \frac{dxdy}{(x^2+y^2)^\beta} = 2\pi[1 - n^{2(\beta-1)}]$$

hence $I$ is convergent for $\beta < 1$.

Similarly, for any compact $L_n = \{(x\ y, z) \in \Sigma_3 : x^2 + y^2 + z^2 \ge \frac{1}{n^2}\}$, $n \in \mathbb{N}^*$, in spherical coordinates we obtain:

$$J_n(\beta) = \iiint_{L_n} \frac{dxdydz}{(x^2+y^2+z^2)^\beta} = \frac{4\pi}{3-2\beta}[1 - n^{2\beta-3}],$$

hence $J$ is convergent for $\beta < \frac{3}{2}$.

**5.** Test for convergence the improper double integral $I = \iint\limits_{\Omega} \ln\sqrt{x^2+y^2}\,dxdy$, where $\Omega = \{(x, y) \in \mathbb{R}^2 : 0 \le x^2 + y^2 \le 1\}$.

Hint. Take $D_n = \{(x, y) \in \Omega : x^2 + y^2 \ge \frac{1}{n^2}\}$, $n \in \mathbb{N}^*$, and use polar coordinates in order to obtain

$$I_n = \iint\limits_{D_n} \ln\sqrt{x^2+y^2}\,dxdy = \frac{\pi}{2}\left[\frac{\ln n}{2n^2} - \frac{1}{n} + \frac{1}{4n^2}\right] \to -\frac{\pi}{2}.$$

**6.** Test for convergence the integrals:

$I = \iint\limits_{\mathbb{R}^2} e^{-\alpha(x^2+y^2)}\cos(x^2+y^2)\,dxdy$, where $\alpha > 0$, and

$J = \iiint\limits_{\Omega} \frac{\ln(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{\alpha}}\,dxdydz$, where $\alpha > 0$, and

$\Omega = \{(x\ y, z) \in \mathbb{R}^3 : 0 < x^2 + y^2 + z^2 \le 1\}$.

Hint. $I$ is convergent for any $\alpha > 0$, since $|\cos(x^2 + y^2)| \le 1$. $J$ is divergent for all $\alpha < \frac{3}{2}$; evaluate it in spherical coordinates.

**7.** Test for convergence the integral $\iint\limits_{\mathbb{R}^2} e^{-x^2-y^2}\frac{x+y}{\sqrt{x^2+y^2}}\,dxdy$ and evaluate it using its principal value.

Hint. $\left|\frac{x+y}{\sqrt{x^2+y^2}}\right| \le 2$, hence we can apply the comparison criterion. The principal value is 0.

**8.** Show that the integral $I = \iint\limits_{\mathbb{R}^2} \frac{\sin\sqrt{x^2+y^2}}{x^2+y^2}\,dxdy$ is divergent.

Hint. The integral is not “absolutely” convergent (see Theorem 3.10), i.e.

$$\iint\limits_{D_n} \frac{|\sin\sqrt{x^2+y^2}\,|}{x^2+y^2}\,dxdy = 2\pi\int_0^n \frac{|\sin r\,|}{r}\,dr \xrightarrow[n\to\infty]{} \infty.$$

However, on particular domains like $D_n = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le n\}$, we have $I_n = \iint\limits_{D_n} \frac{\sin\sqrt{x^2+y^2}}{x^2+y^2}\,dxdy = 2\pi\int_0^n \frac{\sin r}{r}\,dr \to \pi^2$.

# CHAPTER VIII. SURFACE INTEGRALS

The surface integrals extend the notion of double integral in the same manner in which the line integrals extend the simple integrals on $\mathbb{R}$. We will consider only surfaces in $\mathbb{R}^3$, many aspects being similar for the higher dimensional case. At the beginning, we have to analyze the notion of surface.

## § VIII.1. SURFACES IN $\mathbb{R}^3$.

From the mathematical point of view, the notion of surface (as well as that of curve) reduces to a class of functions, which represent different parameterizations. From the practical point of view, the curves and the surfaces are particular objects (sets) in $\mathbb{R}^3$ and $\mathbb{R}^2$, the problem of finding the most adequate parameterization is of capital importance in calculus.

**1.1. __Definition.__** We say that the set $S \subset \mathbb{R}^3$ is a *surface* iff it is the image of a domain (usually open and connected, but sometimes closed!), $D \subseteq \mathbb{R}^2$ through a function $\varphi : D \to \mathbb{R}^3$, called *parameterization* of $S$, i.e. $S = \varphi(D)$. More precisely, any parameterization is a vector function of two variables and three components, i.e. for each $(u, v) \in D$, we note the parameterization by $\varphi(u, v) = (x(u, v), y(u, v), z(u, v)) \in \mathbb{R}^3$, so that the surface becomes

$$S = \{(x(u, v), y(u, v), z(u, v)) \in \mathbb{R}^3, (u, v) \in D\}$$

Their specific classes of parameterizations describe the different types of surfaces.

**1.2. __Types of surfaces.__** We say that the surface $S$ is *simple* iff its parameterization $\varphi$ is 1 : 1. Similarly, $S$ is called *smooth* (*continuous*, *Lipschitzean*, etc.) iff $\varphi \in C^1_{\mathbb{R}^3}(D)$ ($\varphi \in C^0_{\mathbb{R}^3}(D)$, $\varphi \in \mathrm{Lip}_{\mathbb{R}^3}(D)$, etc.).

A smooth surface $S$ is said to be *non-singular*, iff the rank of *the Jacobian matrix* of its parameterization $\varphi$ is equal to two, i.e.:

$$\text{rank}\, \boldsymbol{J}_{\varphi}(u, v) = \text{rank} \begin{pmatrix} \dfrac{\partial x}{\partial u}(u,v) & \dfrac{\partial y}{\partial u}(u,v) & \dfrac{\partial z}{\partial u}(u,v) \\ \dfrac{\partial x}{\partial v}(u,v) & \dfrac{\partial y}{\partial v}(u,v) & \dfrac{\partial z}{\partial v}(u,v) \end{pmatrix} = 2.$$

**1.3. __Remark.__** In this chapter we will consider only simple, smooth and non-singular surfaces, which will be called *regular*. Because each surface admits more parameterizations, one of the fundamental problems in the study of surfaces is to find the *intrinsic properties*, i.e. those properties, which are independent of parameterization. More exactly, a property of a regular surface $S$ is considered intrinsic iff it is maintained by any change of parameterization realized by a diffeomorphism of strict positive Jacobian. It is clear that the precise meaning of this notion is obtained by defining the class of "equivalent" parameterizations.

**1.4. Definition.** Let $\varphi : D \to \mathbb{R}^3$ and $\psi : H \to \mathbb{R}^3$ be two parameterizations of the same surface $S$ in $\mathbb{R}^3$. We say that $\varphi$ and $\psi$ are *equivalent* and we note $\varphi \sim \psi$, iff there exists a diffeomorphism $T : H \to D$ of components

$$\begin{cases} u = \alpha(a,b) \\ v = \beta(a,b) \end{cases}, (a,b) \in H$$

such that $\psi = \varphi \circ T$, and

$$\text{Det}\, \boldsymbol{J}_T = \begin{vmatrix} \dfrac{\partial\alpha}{\partial a}(a,b) & \dfrac{\partial\beta}{\partial a}(a,b) \\ \dfrac{\partial\alpha}{\partial b}(a,b) & \dfrac{\partial\beta}{\partial b}(a,b) \end{vmatrix} > 0$$

at any $(a, b) \in H$. The diffeomorphism $T$ is also called *change of parameters* on the surface $S$.

**1.5. Remarks.** (i) It is easy to verify that ~ is in fact an equivalence. To be more rigorous, we identify the surface $S$ with its class of equivalent parameterizations.

(ii) When we have a parameterization of a surface $S$, we consider that $S$ is explicitly given. There are many practical cases when the surface is described by a condition of the form

$$\Phi(x, y, z) = 0,$$

which is called *implicit* equation of the surface. The problem of finding an explicit form (equation), i.e. to write $(x, y, z) = \varphi(u, v)$ can be generally solved only locally, using the implicit function theorem.

(iii) A particular, but very convenient parameterization of a surface $S \subset \mathbb{R}^3$ is expressed by a function $z = f(x, y)$. More exactly, $D = Pr_{x,y}(D)$, and $f : D \to \mathbb{R}^3$ stands for the parameterization $\varphi(x, y) = (x, y, f(x, y))$.

**1.6. The tangent plane.** If $(u_0, v_0) \in D$, then the corresponding point $M_0 = \varphi(u_0, v_0) \in S$ may be also specified by its *position vector*

$$\vec{r} = x(u_0, v_0)\,\vec{i} + y(u_0, v_0)\,\vec{j} + z(u_0, v_0)\,\vec{k},$$

where $\{\vec{i}, \vec{j}, \vec{k}\}$ is the canonical base of $\mathbb{R}^3$.

The curve

$$\gamma_{u=u_0} = \{\varphi(u_0, v) : (u_0, v) \in D\}$$

is called *curve of parameter* $v$ on $S$ (or coordinate curve of type $u$–constant). Similarly,

$$\gamma_{v=v_0} = \{\varphi(u, v_0) : (u, v_0) \in D\}$$

is called *curve of parameter u* on $S$ (respectively, curve of type $v$–constant). Obviously, $\varphi|_{D[u_0]}$ is a parameterization of $\gamma_{u=u_0}$, while $\varphi|_{D[v_0]}$ is a parameterization of $\gamma_{v=v_0}$, where

$$D[u_0] = \{v \in \mathbb{R} : (u_0, v) \in D\}$$

is the section of $D$ at $u_0$, and similarly,

$$D[v_0] = \{ u \in \mathbb{R} : (u, v_0) \in D\}$$

is the section of $D$ at $v_0$.

The vectors (which are well defined for regular surfaces)

$$\vec{r}_u = \frac{\partial x}{\partial u}\vec{i} + \frac{\partial y}{\partial u}\vec{j} + \frac{\partial z}{\partial u}\vec{k}, \text{ and}$$

$$\vec{r}_v = \frac{\partial x}{\partial v}\vec{i} + \frac{\partial y}{\partial v}\vec{j} + \frac{\partial z}{\partial v}\vec{k}$$

represents the tangent vectors to the curves of coordinates $u$, respectively $v$, at the current point $(x, y, z) = \varphi(u, v) \in S$.

Since $S$ is non-singular, the *normal* vector

$$\vec{n} = \vec{r}_u \times \vec{r}_v$$

is defined at any point of the surface. Using it, the *tangent plane* of the surface is defined by $(\vec{r} - \vec{r}_0) \perp \vec{n}$, i.e.

$$\begin{vmatrix} x - x_0 & y - y_0 & z - z_0 \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \end{vmatrix} = 0.$$

Even if the vectors $\vec{r}_u$ and $\vec{r}_v$ depend on parameterization, the tangent plane is uniquely determined at each point of a regular surface.

**1.7. <u>Proposition.</u>** The tangent plane to $S$ at $M_0$ does not depend on parameterization.

<u>Proof.</u> Let $\varphi \sim \psi$ be two parameterizations of $S$, and let $\vec{n}_\varphi(M_0)$ and $\vec{n}_\psi(M_0)$ be the vectors normal to $S$ at $M_0 \in S$, expressed by the parameterizations $\varphi$ and $\psi$. A direct calculation shows that

$$\vec{n}_\varphi(M_0) = k\, \vec{n}_\psi(M_0),$$

where $k = \mathrm{Det}\, \boldsymbol{J}_T \neq 0$, i.e. $\vec{n}_\varphi(M_0) \| \vec{n}_\psi(M_0)$.

**1.8. <u>Corollary.</u>** (i) If $S$ admits a parameterization $z = f(x, y)$ on its $xy$–projection, then the normal to $S$ has the components $\vec{n} = (-p, -q, +1)$, where $p = f'_x$, and $q = f'_y$. For convenience, if the sense of $\vec{n}$ doesn't matter, i.e. the surface is non-orientated, then we can take $\vec{n} = (p, q, -1)$.

(ii) If $S$ is implicitly defined by the equation $\Phi(x, y, z) = 0$, then the normal takes the form $\vec{n} = (\Phi'_x, \Phi'_y, \Phi'_z)$ since $p = -\dfrac{\Phi'_x}{\Phi'_z}$, $q = -\dfrac{\Phi'_y}{\Phi'_z}$.

The equations of the tangent plane at $M_0(x_0, y_0, z_0) \in S$ will be

$$z - z_0 = p(x - x_0) + q(y - y_0),$$

respectively,

$$(x - x_0)\,\Phi'_x + (y - y_0)\,\Phi'_y + (z - z_0)\,\Phi'_z = 0.$$

The proof reduces to a simple calculation and will be omitted.

Another useful notion in the study of a surface is that of area, which is introduced by the following construction:

**1.9. Definition.** Let $S$ be a regular surface of equation $z = f(x, y)$, where the domain $D = Pr_{x,y}(S)$ of $f$ is a measurable compact domain (*m.c.d.*). To any division $\delta = \{D_1, ..., D_n\}$ of $D$ in *m.c.d.*, we attach a division

$$\Sigma_\delta = \{S_1, ..., S_n\}$$

of $S$, where, for all $i = \overline{1,n}$, we have

$$S_i = \{(x, y, f(x, y)) \in S : (x, y) \in D_i\}.$$

In each sub-domain $D_k$ we choose a point $(x_k, y_k) \in D_k$, so that

$$M_k(x_k, y_k, f(x_k, y_k)) \in S_k$$

for all $k = 1, ..., n$, and we note by $\pi_k$ the tangent plane to $S$ at $M_k$. In each such tangent plane we delimitate a domain

$$T_k = \{(x, y, z) \in \pi_k : (x, y) \in D_k\}, k = 1, ..., n.$$

which is measurable (i.e. it has an area) as image of a *m.c.d.* $D_k$ through $Pr^{-1}_{xy}$. Let $a(T_k)$ denote the area of $T_k$, for all $k = 1, ..., n$.

We say that $S$ *has an area* (is *measurable*, etc.) iff there exists the (finite) number

$$\mathscr{A} = \lim_{\|\delta\| \to 0} \sum_{k=1}^{n} a(T_k),$$

which is the same for all sequences of divisions for which $\|\delta\| \to 0$, and for all possible choices of this "intermediate" points $M_k \in S_k$, $k = 1, ..., n$. In this case we note $\mathscr{A} = a(S)$, and we call it *area* of $S$.

For the evaluation of the area of a surface we mention:

**1.10. Theorem.** Let $S$ be a regular surface for which $D = Pr_{xy}S$ is a *m.c.d.* in $\mathbb{R}^2$, and $z = f(x, y)$, where $f : D \to \mathbb{R}$ is the equation of $S$. Then $S$ has area and it is expressed by the double integral

$$a(S) = \iint_D \sqrt{1 + f_x'^2 + f_y'^2}\, dxdy. \qquad (1)$$

Proof. Let $\theta_k$ be the angle between the $oz$ axes and the normal $\vec{n}_k$ at the point $M_k \in S_k \subset S$. Using $\theta_k$, can specify the relation between the area $a(D_k)$ of $D_k$ and that of $T_k$, namely

$$a(D_k) = a(T_k) \cos \theta_k.$$

We may find of the value of $\cos \theta_k$ from the formula $\vec{n}_k \vec{k} = \|\vec{n}_k\| \|\vec{k}\| \cos \theta_k$, while gives $\cos \theta_k = [1 + f_x'^2(x_k, y_k) + f_y'^2(x_k, y_k)]^{-1/2}$ for all $k = 1, ..., n$. Consequently,

$$\sum_{k=1}^{n} a(T_k) = \sum_{k=1}^{n} \sqrt{1 + f_x'^2(x_k, y_k) + f_y'^2(x_k, y_k)} \cdot a(D_k)$$

has the form of a Riemannian sum of a double integral on $D$, as mentioned in the theorem. The existence of this integral is assured by theorem 7, §2, chapter VII, since $f$ has continuous partial derivatives on $D$, and $D$ is a *m.c.d.*

The area of a surface may be expressed by other formulas which make use of some specific notations. More exactly, if $\varphi : H \to \mathbb{R}^3$ is a parameterization of $S$, of components $x(u, v)$, $y(u, v)$, $z(u, v)$, and if we note

$$A = \frac{D(y,z)}{D(u,v)},\ B = \frac{D(z,x)}{D(u,v)},\ C = \frac{D(x,y)}{D(u,v)}$$

then the normal becomes $\vec{n} = A\vec{i} + B\vec{j} + C\vec{k}$, and

$$\| \vec{r}_u \times \vec{r}_v \| = \sqrt{A^2 + B^2 + C^2}$$

holds at any point $M \in S$.

Other useful notations are the *Gauss* coefficients:

$$E = \|\vec{r}_u\|^2 = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2$$

$$F = \vec{r}_u \cdot \vec{r}_v = \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v}$$

$$G = \|\vec{r}_v\|^2 = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2$$

A direct computation shows that $A^2 + B^2 + C^2 = EG - F^2$.

**1.11. <u>Corollary.</u>** Let $S$ be a smooth surface for which $D = Pr_{xy}(S)$ is a *m.c.d.*, and let $z = f(x, y)$ be the equation of $S$. If $\varphi : H \to \mathbb{R}^3$ is another parameterization of $S$, then the following formulas hold:

$$a(S) = \iint_H \sqrt{A^2 + B^2 + C^2}\, dudv \qquad (2)$$

$$a(S) = \iint_H \| \vec{r}_u \times \vec{r}_v \|\, dudv \qquad (3)$$

$$a(S) = \iint_H \sqrt{EG - F^2}\, dudv \qquad (4)$$

<u>Proof.</u> Let $T : H \to D$ be a transformation (diffeomorphism) of components $x = \alpha(u, v)$ and $y = \beta(u, v)$, which relates the parameterizations. More exactly, $(x, y, z) = \varphi(u, v)$ means

$$\begin{cases} x = \alpha(u,v) \\ y = \beta(u,v) \\ z = f(\alpha(u,v), \beta(u,v)) \end{cases}$$

for all $(u, v) \in H$. Using the partial derivatives of $z$,

$$\frac{\partial z}{\partial u} = f_x^{/} \frac{\partial \alpha}{\partial u} + f_y^{/} \frac{\partial \beta}{\partial u}$$
$$\frac{\partial z}{\partial v} = f_x^{/} \frac{\partial \alpha}{\partial v} + f_y^{/} \frac{\partial \beta}{\partial v}$$

we obtain $f_x^{/} = \frac{A}{C}$, and $f_y^{/} = \frac{B}{C} f'_y$. Changing the variables $(x, y) \mapsto (u, v)$ in (1) we obtain

$$a(S) = \iint_D \sqrt{1 + f_x^{/2} + f_y^{/2}}\; dxdy = \iint_E \sqrt{A^2 + B^2 + C^2} \cdot \frac{1}{C} \text{Det}\, \boldsymbol{J}_T\; dudv,,$$

which represents (2), since $C = \text{Det}\, \boldsymbol{J}_T$.

Formula (3) is a simple transformation of (2) because

$$\vec{r}_u \times \vec{r}_v = A\vec{i} + B\vec{j} + C\vec{k}.$$

Finally, (4) follows from (3) as a consequence of the identity

$$\| \vec{r}_u \times \vec{r}_v \|^2 = \| \vec{r}_u \|^2 \| \vec{r}_v \|^2 - [\, \vec{r}_u \;\; \vec{r}_v \,]^2 . \qquad \diamondsuit$$

More than the equivalence of the formulas (1), (2), (3) and (4), the area of a surface is an intrinsic characteristic of the surface, i.e. it is the same for all equivalent parameterizations.

**1.12. <u>Theorem.</u>** If $S$ is a regular surface which has a parameterization on the *m.c.d.* $D = Pr_{xy}(S)$, then any other equivalent parameterization of $S$ gives the same value for the area of $S$.

<u>Proof.</u> Let $\varphi : H \to \mathbb{R}^3$ and $\psi : L \to \mathbb{R}^3$ be equivalent parameterizations of the regular surface $S$, of components

$$\varphi(u, v) = (x(u, v), y(u, v), z(u, v)), \text{ and}$$
$$\psi(a, b) = (\tilde{x}(a, b), \tilde{y}(a, b), \tilde{z}(a, b)),$$

and let $A$, $B$, $C$, respectively $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ be the corresponding coefficients. According to the above corollary, both double integrals

$$\iint_H \sqrt{A^2 + B^2 + C^2}\, dudv, \text{ and}$$
$$\iint_L \sqrt{\tilde{A}^2 + \tilde{B}^2 + \tilde{C}^2}\, dadb$$

represent the same

$$a(S) = \iint_D \sqrt{1 + f_x^{/2} + f_y^{/2}}\; dxdy.$$

We mention that the proof could be based on the relations $\tilde{A} = A\Delta;$ $\tilde{B} = B\Delta$; $\tilde{C} = C\Delta$, where $\Delta = \text{Det}\, \boldsymbol{J}_V$, and $V : L \to H$ is a diffeomorphism which realizes the change of parameters $(a, b) \mapsto (u, v)$. $\diamondsuit$

**PROBLEMS § VIII.1.**

**1.** Find the area of the triangle cut out by the coordinate planes from the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, where $a, b, c \in \mathbb{R}^+$.

Hint. The $xy$–projection of $S$ in $D =\{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, \frac{x}{a}+\frac{y}{b}\leq 1\}$ and the equation of the surface has the form $z = f(x, y)$, where

$$f(x, y) = c(1-\frac{x}{a}-\frac{y}{b}).$$

Consequently, $a(S) = \iint_D \sqrt{1+f_x'^2+f_y'^2}\, dxdy = \frac{1}{2}\sqrt{a^2b^2+b^2c^2+c^2a^2}$ .

**2.** Compute the area of the helicoidal surface of polar equations $\begin{cases} x = r\cos t \\ y = r\sin t \\ z = kt \end{cases}$, $0 \leq t \leq \frac{\pi}{2}$, $0 < a \leq r \leq b$, where $a, b, k \in \mathbb{R}^*_+$.

Hint. $Pr_{xy}(S) = \{(x, y) \in \mathbb{R}^2 : a^2 \leq x^2 + y^2 \leq b^2, x \geq 0, y \geq 0\}$. Evaluate the double integral which represents the area in polar coordinates.

**3.** Let $\mathscr{C}$ be the cylinder of equation $x^2 + y^2 = ax$, and let $\mathscr{S}$ be the sphere of equation $x^2 + y^2 + z^2 = a^2$. Evaluate the area $a(S)$ if:
(i) $S$ is that part of $\mathscr{C}$ which is cut out by $\mathscr{S}$
(ii) $S$ is the part of $\mathscr{S}$ inside $\mathscr{C}$ .
Hint. (*i*) Consider $y = f(x, z)$ on $Pr_{xz}(S) = \{(x, z) \in \mathbb{R}^2 : 0 \leq x^2 + z^2 \leq a, x > 0\}$.
(*ii*) Take $z = f(x, y)$ on $Pr_{xy}(S) = \{(x, y) \in \mathbb{R}^2 : (x - (a/2))^2 + y^2 \leq a^2/4\}$ .

**4.** Calculate the area of the torus obtained by rotating the circle of center $(R, 0,0)$ and radius $r$, where $0 < r < R$, lying in the $xoy$–plane, around the $oy$–axis.
Hint. $S$ is the image of $D = \{(u, v) \in \mathbb{R}^2 : 0 \leq u, v \leq 2\pi\}$ through $\varphi$ of components $x = (R + r\cos u)\cos v$, $y = r \sin u$, and $z = (R + r\cos u)\sin v$, hence $\| \vec{r}_u \times \vec{r}_v \| = r(R + r\cos u)$, and $a(S) = 4\pi^2 Rr$.

**5.** Compare the areas of the parts of a paraboloid $x^2 + y^2 = 2az$ (circular) and $x^2 - y^2 = 2az$ (hyperbolic), cut out by the cylinder $x^2 + y^2 = R^2$.
Hint. Use polar coordinates. The areas are equal.

**6.** Find the area of an ellipsoid of half axes $a$, $b$, $c$.
Hint. $S = \varphi(D)$, where $D = \{(u, v) \in \mathbb{R}^2 : u \in [0, \pi],\ v \in [0, 2\pi]\}$, and $\varphi(u, v) = (a \sin u \cos v, b \sin u \cos v, c \cos u)$. Use the formula

$$a(S) = \iint_D \sqrt{A^2 + B^2 + C^2}\, du dv.$$

**7.** (**Schwartz's example**) Let $S$ be the lateral surface of a cylinder of radius $r$ and altitude $h$. By dividing $h$ into $n^3$ equal parts, $n \in \mathbb{N}$, using planes parallel to the bases, we obtain $n^3 + 1$ circles $C_0, C_1, ..., C_{n^3}$ on $S$. On $C_0$ we consider $2n$ equidistant points. The generators corresponding to these points meet the other circles in $2n$ points denoted with similar indices. Now, from each circle $C_k$ we delete the points with even indices if $k$ is odd, and the points with odd indices if $k$ is even. Each pair of remaining successive points on the same circle and the closest point of a neighboring circle determine a triangle $\Delta$. Evaluate the area $a(\Delta)$, show that the sum of all these areas tends to $\infty$ when $n \to \infty$, and explain why this sum does not approximate $a(S)$ .
Hint. There are $2u \cdot u^3 = 2u^4$ such triangles of areas

$$a(\Delta) = \frac{1}{2} 2(\sin\frac{\pi}{n}) r \sqrt{r^2 (1 - \cos\frac{\pi}{n})^2 + \left(\frac{h}{n^3}\right)^2} > kn^{-3}$$

for some $k > 0$. The explanation consists in making evident the different directions of the normal vectors to $\Delta$ and to $S$ (see also [NS] vol. II).

## § VIII.2. FIRST TYPE SURFACE INTEGRALS

Similarly to the line integrals of the first type, the surface integral of the first type refers to scalar functions defined on domains, which contain the surface. They are useful in evaluating the mass of a lamina, its gravity center, inertia moments, or forces of interaction.

**2.1. <u>The construction of the integral sums.</u>** Let $S$ be a regular surface of parameterization $\varphi: D \to \mathbb{R}^3$, where $D$ is a *m.c.d.* in $\mathbb{R}^2$. Let also $U : \Omega \to \mathbb{R}$ be a bounded scalar function, where $\Omega$ is a domain in $\mathbb{R}^3$ which contains $S$ (it is sometimes sufficient to ask $U : S \to \mathbb{R}$, as for example when $U$ is the density of the material surface $S$). If $\delta = \{D_1, ..., D_n\}$ is a partition of $D$, then we consider the subsequent partition $\Sigma_\delta = \{S_1, ..., S_n\}$ of $S$, and a system $\mathscr{S} = \{M_k \in S_k : k = 1, ..., n\}$ of "intermediate" points, exactly as for evaluating the area of $S$. Then the sums

$$\sigma_U(\delta, \mathscr{S}) = \sum_{k=1}^{n} U(M_k)\, a(S_k)$$

represent the *integral sums of the first type* of $U$ on $S$ .

**2.2. <u>Definition.</u>** We say that $U$ is *integrable* on $S$ iff there exists (finite)

$$I = \lim_{\|\delta\| \to 0} \sigma_U(\delta, \mathscr{S}),$$

independently of the sequence of divisions for which $\|\delta\| \to 0$, and independently of the systems of intermediate points. More exactly, for any $\varepsilon > 0$, there exists $\eta > 0$ such that

$$| \sigma_U(\delta, \mathscr{S}) - I | < \varepsilon$$

holds whenever $\|\delta\| < \eta$, and for arbitrary $\mathscr{S}$. In this case we say that $I$ is the *surface integral* of $U$ (of the first type), and we note

$$I = \iint_S U(x, y, z)dS ,$$

or alternatively

$$I = \iint_S UdS = \iint_S Ud\mu \text{, etc.}$$

One of the fundamental problems is to specify classes of integrable functions, and methods of evaluating the integrals.

**2.3. <u>Theorem.</u>** Let $S$ be a regular surface, and $U : S \to \mathbb{R}$ be a continuous function. Then, $U$ is integrable on $S$, i.e. there exists the surface integral of the first type of $U$ on $S$, and

$$\iint_S UdS = \iint_D (U \circ \varphi)(u,v)\sqrt{A^2(u,v) + B^2(u,v) + C^2(u,v)}\; dudv \qquad (1)$$

where $\varphi : D \to \mathbb{R}^3$ is a parameterization of $S$.

Proof. Because $D$ is a *m.c.d.* and $V = (U \circ \varphi)\sqrt{A^2 + B^2 + C^2}$ is continuous, there exists the double integral in the right side of the claimed relation. Let $\delta = \{D_1, ..., D_n\}$ be a division of $D$. Using the mean-value theorem for double integrals we obtain

$$a(S_k) = \iint_{D_k} \sqrt{A^2 + B^2 + C^2}\,dudv =$$

$$= \sqrt{A^2(\tilde{u}_k, \tilde{v}_k) + B^2(\tilde{u}_k, \tilde{v}_k) + C^2(\tilde{u}_k, \tilde{v}_k)}\,a(D_k)$$

where $(\tilde{u}_k, \tilde{v}_k) \in D_k$. Consequently, considering an arbitrary system of intermediate points $\mathscr{S} = \{ \varphi(u_k, v_k) : k = 1, ..., n\}$, the integral sums take the form

$$\sigma_U(\delta, \mathscr{S}) = \sum_{k=1}^{n} U(\varphi(u_k, v_k))a(S_k) =$$

$$= \sum_{k=1}^{n} (U \circ \varphi)(u_k, v_k) \sqrt{A^2(\tilde{u}_k, \tilde{v}_k) + B^2(\tilde{u}_k, \tilde{v}_k) + C^2(\tilde{u}_k, \tilde{v}_k)}\,a(D_k).$$

On the other hand, since $V$ is integrable on $D$, for every $\varepsilon > 0$ there exists $\eta > 0$ such that for $\|\delta\| < \eta$, and for arbitrary $\mathscr{S}$, we have

$$\left| \iint_D V dudv - \sum_{k=1}^{n} (U \circ \varphi)(u_k, v_k) a(D_k) \right| < \frac{\varepsilon}{2}.$$

Now, we can evaluate

$$\left| \iint_D V dudv - \sigma_U(\delta, \mathscr{S}) \right| \le \left| \iint_D V dudv - \sum_{k=1}^{n} (U \circ \varphi)(u_k, v_k) a(D_k) \right| +$$

$$+ \left| \sum_{k=1}^{n} (U \circ \varphi)(u_k, v_k) a(D_k) - \sum_{k=1}^{n} (U \circ \varphi)(u_k, v_k) \sqrt{A^2 + B^2 + C^2}(\tilde{u}_k, \tilde{v}_k) \right| < \varepsilon.$$

The last modulus is less than $\frac{\varepsilon}{2}$, since $|U \circ \varphi|$ is bounded on $D$, and $\sqrt{A^2 + B^2 + C^2}$ is uniformly continuous on $D$. ◇

When defining the integral sums, the values of $U$ on $S$, and the areas of the surfaces $S_k$ do not depend on parameterizations, hence the surface integral is uniquely defined by $S$ and $U$. In fact:

**2.4. Corollary.** The surface integral of the first type does not depend on parameterization.

Proof. Let $\psi : H \to \mathbb{R}^3$ be another parameterization of $S$ in the above theorem, and let $T : H \to D$ be the diffeomorphism for which $\psi = \varphi \circ T$.

Changing the variables $(u, v) = T(a, b)$ in the double integral (1), we obtain

$$\iint_S UdS = \iint_H (U \circ \varphi \circ T)(a,b)\sqrt{\tilde{A}^2 + \tilde{B}^2 + \tilde{C}^2}(a,b)\ dadb .$$

Because $\tilde{A} = A\Delta$, $\tilde{B} = B\Delta$, $\tilde{C} = C\Delta$, where $\Delta = \mathrm{Det}\, \boldsymbol{J}_T(a, b)$, we obtain

$$\iint_S UdS = \iint_H [(U \circ \psi)\sqrt{\tilde{A}^2 + \tilde{B}^2 + \tilde{C}^2}](a,b)\ dadb ,$$

i.e. different equivalent parameterizations of the surface give the same value of the surface integral of the first type. ◇

**2.5. Corollary.** Using the notations in § VIII.1, the surface integral can be expressed by the formula

$$\iint_S UdS = \iint_D U(x, y, f(x, y))\sqrt{1 + p^2 + q^2}dxdy , \qquad (2)$$

where $z = f(x, y)$ is the equation of $S$, $f : D \to \mathbb{R}$, $p = f_x^{/}$, $q = f_y^{/}$. Other forms of the same integral are

$$\iint_S UdS = \iint_D [U \circ \varphi)\, \| \vec{r}_u \times \vec{r}_v \|](u,v)\ dudv, \text{ and} \qquad (3)$$

$$\iint_S UdS = \iint_D [(U \circ \varphi)\sqrt{EG - F^2}](u,v)\ dudv . \qquad (4)$$

In fact, according to § VIII.1, where we have expressed the element of area in several forms, we have seen that

$$\sqrt{1 + p^2 + q^2} = \sqrt{A^2 + B^2 + C^2} = \| \vec{r}_u \times \vec{r}_v \| = \sqrt{EG - F^2} .$$

**2.6. Remark.** So far, we have used the *xy*–projection to study the surfaces and the surface integrals of the first type. Similar results may be obtained for *yz*, or *zx*–projections. In practice, we can divide the given surface into a finite number of surfaces, which admit such projections. This decomposition is frequently necessary if the equation of the surface is implicit.

The general properties of the first type surface integrals are common to other types of integrals, namely.

**2.7. Proposition.** The surface integral of the first type is:

(i) *linear* relative to the function, i.e.

$$\iint_s (\alpha U + \beta V)dS = \alpha \iint_S UdS + \beta \iint_S VdS ;$$

(ii) *additive* relative to the surface, i.e.

$$\iint_{S_1 \cup S_2} UdS = \iint_{S_1} UdS + \iint_{S_2} UdS ,$$

where $S_1$, $S_2$ are regular surfaces without common interior points.

The proof is a simple reduction to the similar properties of the double integral, and will be omitted.

**PROBLEMS § VIII.2**

**1.** Evaluate the surface integral

$$I = \iint_S (x+y+z)dS$$

where $S$ is the surface of the cube $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$ .
Hint. The integral on the faces $z = 0$ and $z = 1$, where $dS = dxdy$ reduces to

$$\int_0^1\int_0^1 (x+y)dxdy + \int_0^1\int_0^1 (x+y+1)dxdy = \int_0^1\int_0^1 (2x+2y+1)dxdy = 3.$$

Similarly, we treat the other pairs of faces, so that $I = 9$.

**2.** Evaluate the integral $\iint_S (xy+yz+zx)dS$ , where $S$ is that part of the cone $z = \sqrt{x^2+y^2}$ , cut out by the surface $x^2+y^2 = 2ax$ .
Answer. $\dfrac{64\sqrt{2}}{15}a^4$.

**3.** Find the mass of a material surface $S$ of equation $z = \dfrac{x^2+y^2}{2}$, $0 \le z \le 1$, which has the local density $\rho(x,y,z) = z$ .
Answer. $\dfrac{2\pi}{15}(1+6\sqrt{3})$.

**4.** Evaluate the moment of inertia of a spherical surface of radius $r$ and of constant density ρ, relative to a diameter.
Hint. $I = \rho\iint_S (x^2+y^2)dS = \rho\dfrac{8}{3}\pi r^4$ . The spherical coordinates are advisable, since $\sqrt{A^2+B^2+C^2} = r^2 \sin\theta$, and $I = 2\pi r^4\rho \int_0^\pi \sin^3\theta d\theta$.

**5.** Calculate the moment of inertia, relative to the $xOy$ plane, of that part of the conic surface $z = \sqrt{x^2+y^2}$ , for which $0 \le z \le 1$, if the local density is $\rho(x,y,z) = 1+xy$ .
Hint. By definition, $I_{xOy} = \iint_S z^2(1+xy)dS = \sqrt{2}\iint_D (x^2+y^2)(1+xy)\,dx\,dy$, where $D = \{(x,y)\in\mathbb{R}^2 : x^2+y^2 \le 1\}$. The result is $I_{xOy} = \dfrac{\pi\sqrt{2}}{2}$ .

**6.** Find the attraction force exerted by an uniform spherical surface on a point-wise mass $m$ located in the interior (exterior) of the sphere.

Hint. $F_z = km\,\rho \iint\limits_S \frac{z-r}{[x^2+y^2+(z-r)^2]^{3/2}} dS$, where $(0, 0, r)$ is the location of $m$, $\rho$ is the density of $S$, and $k$ is depending on units. Using the spherical coordinates $x = R \sin\theta \cos\varphi$, $y = R \sin\theta \sin\varphi$, $z = R \cos\theta$, where $R$ is the radius of the sphere, we obtain $F_z = 2\pi km\,\rho[R^3 I - rR^2 J]$, and

$$I = \int_0^\pi \frac{\cos\theta \sin\theta}{[R^2 - 2rR\cos\theta + r^2]^{3/2}} d\theta = \frac{2r}{R^2(R^2 - r^2)} \text{ (by parts), and}$$

$$J = \int_0^\pi \frac{\sin\theta}{[R^2 - 2rR\cos\theta + r^2]^{3/2}} d\theta = \frac{2}{R(R^2 - r^2)}.$$

Consequently, $F_z = 0$. Because of symmetry, we have $F_x = F_y = 0$ too.

**7.** Find the potential created at (0, 0, 0) by an electric change of density $\rho(x, y, z) = 2 - \frac{a}{\sqrt{x^2+y^2}}$, distributed on a conical surface of equation

$$0 \le z = a - \sqrt{x^2+y^2}, \; a > 0.$$

Hint. The potential is $\Phi = k \iint\limits_S \frac{\rho(x,y,z)dS}{\sqrt{x^2+y^2+z^2}}$, where $k$ depends on units. In particular, $dS = \sqrt{2}\,dxdy$. The surface integral can be reduced to a double integral on $D = Pr_{xy}(S)$, which can be easily evaluated in polar coordinates. The searched potential is $\Phi = 0$.

## § VIII.3. SECOND TYPE SURFACE INTEGRALS

In this section we study the surface integral of a vector function, which defined on the surface. In order to explain the meaning of this integral, we start out with an example:

**3.1. Example.** (The flux of an incompressible liquid through a surface) Let us consider that the domain $\Omega \subset \mathbb{R}^3$ is full of liquid, which is in stationary movement. To describe this movement we use the so-called *vector field of speeds*, $\vec{V} : \Omega \to \mathbb{R}^3$, which defines the velocity

$$\vec{V}(x, y, z) = (V_1(x, y, z), V_2(x, y, z), V_3(x, y, z))$$

at each point $(x, y, z) \in \Omega$ (not depending on time since the movement is stationary). Now let $S \subset \Omega$ be a (regular) surface, for which we need to determine the quantity of liquid, which is passing over the surface in the unit of time (also called *flux*). Obviously, evaluating this quantity supposes a sense of the normal vector at each point of the surface, such that specifying what "comes in" and what "goes out" to be possible (see the orientated surfaces below).

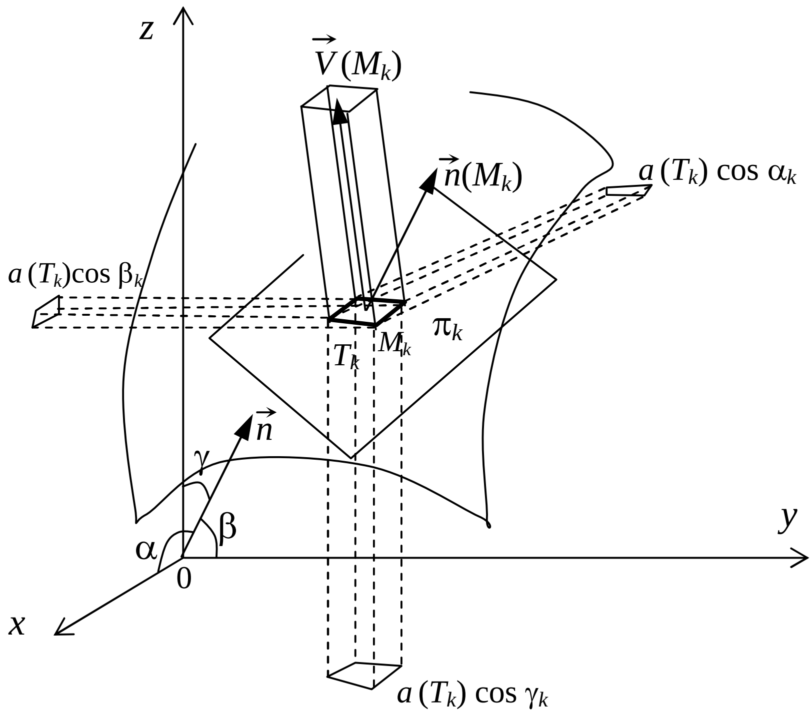



Fig. VIII.3.1.

If we refer to a small part $S_k \subset S$, or to its corresponding approximation $T_k$ of the tangent plane $\pi_k$ at $M_k \in S_k$, the seek quantity is contained in the volume $v_k$ of a parallelepiped of basis $T_k$ and side $\vec{V}(M_k)$.

More exactly (see Fig. VIII.3.1), since $\vec{n}$ $(M_k) \perp \pi_k$, we have

$$v_k = <\vec{V}(M_k), \vec{n}(M_k)> area(T_k) = (\vec{V} \cdot \vec{n})(M_k)\, a\,(T_k).$$

If $S_k$ is an element of the partition $\delta = \{S_1, ..., S_n\}$ of $S$, and $M_k \in S_k$ is an intermediate point of the system $\mathscr{S} = \{M_1, ..., M_n\}$, then

$$v(\delta, \mathscr{S}) = \sum_{k=1}^{n} v_k = \sum_{k=1}^{n} (\vec{V} \cdot \vec{n})(M_k) a(T_k)$$

represents an approximation of the sought volume. Further on, if

$$\vec{n} = \cos\alpha\, \vec{i} + \cos\beta\, \vec{j} + \cos\gamma\, \vec{k}$$

is the unit normal, then we can explicit the scalar product $(\vec{V} \cdot \vec{n})$, and we obtain

$$v(\delta, \mathscr{S}) = \sum_{k=1}^{n} (V_1 \cos\alpha + V_2 \cos\beta + V_3 \cos\gamma)(M_k) a(T_k) =$$

$$= \sum_{k=1}^{n} [V_1(M_k) a(\mathrm{Pr}_{yz}(T_k)) + V_2(M_k) a(\mathrm{Pr}_{zx}(T_k)) + V_3(M_k) a(\mathrm{Pr}_{xy}(T_k))].$$

Because generally speaking, better approximations correspond to finer partitions of the surface, it is natural to define the flux of $\vec{V}$ through $S$ as

$$v = \lim_{\|\delta\| \to 0} v(\delta, \mathscr{S}).$$

This example shows that before defining the general notion of surface integral of second type, we must clarify the meaning of *orientation* on a surface (compare to the orientation of a curve in § VI.1).

**3.2. <u>Orientated surfaces.</u>** As usually, an explicit writing of the above formulas supposes some parameterization $\varphi : D \to \mathbb{R}^3$ of $S$, when $D \subseteq \mathbb{R}^2$ is a measurable compact domain of the plane. According to the definitions in § VIII.1, $S$ is regular means that $\varphi$ is 1 : 1, of class $C^1$, and rank $\boldsymbol{J}_\varphi = 2$ on $D$. More exactly, $S$ is defined by a class of such equivalent parameterizations, where $\varphi \sim \psi$ denotes the existence of a diffeomorphism $T$ between the domains of $\varphi$ and $\psi$ such that Det $\boldsymbol{J}_T \neq 0$. Consequently, either Det $\boldsymbol{J}_T > 0$, or Det $\boldsymbol{J}_T < 0$, i.e. the class of all parameterizations can be split into two subclasses, each of them consisting of those parameterizations which are related by a "positive" diffeomorphism (Det $\boldsymbol{J}_T > 0$). To *orientate* the surface $S$ means to chose one of these subclasses of parameterizations as defining the positive sense of the normal at each point of $S$. These considerations are based on the following:

**3.3. <u>Proposition.</u>** Let $\varphi : D \to \mathbb{R}^3$ and $\psi : E \to \mathbb{R}^3$ be parameterizations of the regular surface $S$, and let $T : E \to D$ be a diffeomorphism for which $\psi = \varphi \circ T$. If $\vec{n}_\varphi(M)$ and $\vec{n}_\psi(M)$ represent the unit normal vectors at $M \in S$, corresponding to these parameterizations, then we have:

(i) $\vec{n}_\varphi(M) = \vec{n}_\psi(M)$ if Det $\boldsymbol{J}_T > 0$ ($T$ is positive)

(ii) $\vec{n}_\varphi(M) = - \vec{n}_\psi(M)$ if Det $\boldsymbol{J}_T < 0$ ($T$ is negative)

Proof. Let $A_\varphi(M)$, $B_\varphi(M)$, $C_\varphi(M)$ and $A_\psi(M)$, $B_\psi(M)$, $C_\psi(M)$ be the differential coefficients corresponding to the parameterizations φ and ψ, at the current point $M \in S$. Consequently, the unit normal vectors, which correspond to these parameterizations, are

$$\vec{n}_\varphi(M) = \frac{A_\varphi \vec{i} + B_\varphi \vec{j} + C_\varphi \vec{k}}{\sqrt{A_\varphi^2 + B_\varphi^2 + C_\varphi^2}}(M)$$

$$\vec{n}_\psi(M) = \frac{A_\psi \vec{i} + B_\psi \vec{j} + C_\psi \vec{k}}{\sqrt{A_\psi^2 + B_\psi^2 + C_\psi^2}}(M).$$

Similarly to theorem 12 in §1 (chapter VIII), from $\psi = \varphi \circ T$ we deduce $A_\psi = A_\varphi\Delta$, $B_\psi = B_\varphi\Delta$, and $C_\psi = C_\varphi\Delta$, where $\Delta = \operatorname{Det} \boldsymbol{J}_T$. ◇

**3.4. Examples.** 1) If $S$ admits a parameterization $z = f(x, y)$ on the projection $D=Pr_{xy}(S)$, then usually, the positive sense of the normal is that for which the angle between $oz$ (i.e. $\vec{k}$) and $\vec{n}$ is in the interval $[0, \frac{\pi}{2}]$.

2) If $S$ is closed, then it divides the space into two parts, namely the *interior* and the *exterior* of $S$. The positive sense of the normal is usually chosen outwards. (However, the exact meaning of orientation and closeness is obtained in much deeper theories, e.g. see [SL], [CI], etc.).

3) When referred to the vectors $\vec{r}_u$ and $\vec{r}_v$, $\vec{n}$ is orientated according to the right-hand screw rule: by rotating the hand of the screw from $\vec{r}_u$ to $\vec{r}_v$, the screw is driven in the positive sense of $\vec{n}$. In this way the orientation of $D$ is carried to $S$.

4) The orientation on $S$ can be referred to the particular sense, which is defined on the frontier of $S$. In this case we apply the same right-hand screw rule.

5) As an example of non-orientated surface we mention the famous Möbius strip. It is obtained from a plane rectangle of sides $l$ and $L$, where $l<<L$, by gluing the smaller sides cross-wide (as sketched in Fig. VIII.3.2).

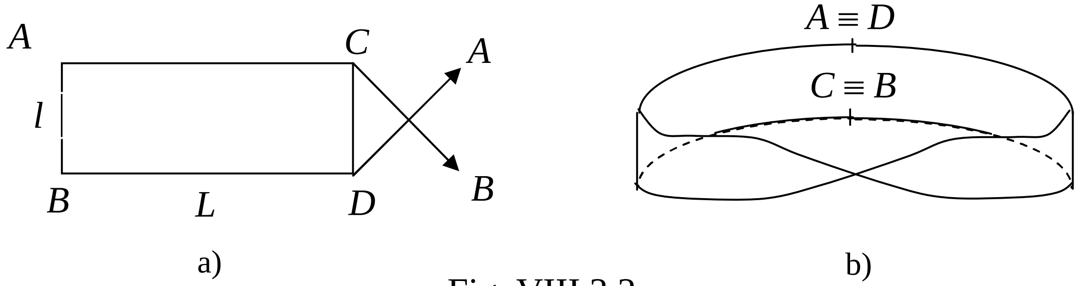



Fig. VIII.3.2

The resulting surface allows no 1:1 parameterization. The coordinates of any point depend on the "face" on which the point is lying, even though we can pass from one face to another without touching the boundary. Therefore we cannot specify a positive sense of the normal at any point of the surface, i.e. the Möbius strip is not orientated.

The surface integral of the second type is defined by analogy to the above notion of flux through an orientated surface:

**3.5. <u>The integral sums.</u>** Let $\vec{V}: \Omega \to \mathbb{R}^3$ be a vector function on the domain $\Omega \subseteq \mathbb{R}^3$, and let $S \subset \Omega$ be a regular orientated surface of parameterization $\varphi : D \to \mathbb{R}^3$, where $D \subseteq \mathbb{R}^2$ is a measurable compact domain (*m.c.d*). If $\delta = \{D_1, ..., D_n\}$ is a partition of $D$, the corresponding division $\Sigma_\delta = \{S_1, ..., S_n\}$ of $S$ consists of parts $S_k = \varphi(D_k) \subset S$. Choosing $M_k \in S_k$ on each $S_k$, $k = 1, ..., n$, we obtain a system of intermediate points $\mathscr{S} = \{M_k : k=1, ..., n\}$. Let $\pi_k$ be the tangent plane at $M_k$ to $S$, and let $T_k$ be the projection of $S_k$ on $\pi_k$, $k = 1,..., n$ . The sum

$$\sigma_{\vec{V},S}(\delta, \mathscr{S}) =$$

$$= \sum_{k=1}^{n} [V_1(M_k) a(\mathrm{Pr}_{yz}(T_k)) + V_2(M_k) a(\mathrm{Pr}_{zx}(T_k)) + V_3(M_k) a(\mathrm{Pr}_{xy}(T_k))]$$

is called *surface integral sum of second type* of $\vec{V}$ on $S$.

**3.6. <u>Definition.</u>** We say that $\vec{V}$ *is integrable* on $S$ iff the above surface integral sums of second type have a limit

$$I = \lim_{\|\delta\| \to 0} \sigma_{\vec{V},S}(\delta, \mathscr{S}),$$

which is independent of the sequence of divisions with $\|\delta\| \to 0$, and of the choice of intermediate points. In this case we note

$$I = \iint_S V_1 dydz + V_2 dzdx + V_3 dxdy,$$

and we say that $I$ is the *surface integral of the second type* of $\vec{V}$ on $S$, also called the *flux* of $\vec{V}$ through $S$.

The following theorem indicates a class of integrable functions.

**3.7. <u>Theorem.</u>** Let $S \subset \Omega$ be a regular orientated surface, and let $\vec{V}: \Omega \to \mathbb{R}^3$ be a vector function. If $\vec{V}$ is continuous on $S$, then:

(i) $\vec{V}$ is integrable on $S$, and

(ii) its surface integral (of the second type) reduces to a surface integral of the first type according to the formula

$$\iint_S V_1 dydz + V_2 dzdx + V_3 dxdy = \iint_S \vec{V} \cdot \vec{n}\, dS .$$

<u>Proof.</u> Because $\vec{V}$ and $\vec{n}$ are continuous on $S$, theorem 3 in §VIII.2 assures the existence of the surface integral of the first type

$$I = \iint_S \vec{V} \cdot \vec{n}\, dS .$$

Consequently, the problem reduces to show that for any $\varepsilon > 0$ there exists $\eta > 0$ such that for any partition $\delta$ and $\mathscr{S}$, for which $\|\delta\| < \eta$, we have

$$|\,\sigma_{\vec{V},S}(\delta, \mathscr{S}) - I\,| < \varepsilon.$$

In fact, because $\vec{V}\cdot\vec{n}$ is continuous there exists $\lambda = \sup\limits_S |\vec{V}\cdot\vec{n}|$. In addition, since $S$ is measurable, there exists $\eta_1 > 0$ such that

$$\left|\sum_{k=1}^{n}\left[a(T_k) - a(S_k)\right]\right| < \frac{\varepsilon}{2\lambda}$$

holds for any division $\delta = \{D_1, ..., D_n\}$ for which $\|\delta\| < \eta_1$. Consequently, for such divisions we have

$$|\,\sigma_{\vec{V},S}(\delta, \mathscr{S}) - \sigma_{\vec{V},\vec{n}}(\delta, \mathscr{S})\,| \le \sum_{k=1}^{n} |(\vec{V}\cdot\vec{n})(M_k)|\,|a(T_k) - a(S_k)| \le \frac{\varepsilon}{2}. \qquad (*)$$

On the other hand, since $\vec{V}\cdot\vec{n}$ is integrable on $S$, there exists $\eta_2 > 0$ such that for $\|\delta\| < \eta_2$, we have

$$|\,\sigma_{(\vec{V}\cdot\vec{n}),S}(\delta, \mathscr{S}) - I\,| < \frac{\varepsilon}{2}. \qquad (**)$$

Combining (*) and (**) , for $\|\delta\| < \eta = min\{\eta_1, \eta_2\}$, we obtain

$$|\,\sigma_{\vec{V},S}(\delta, \mathscr{S}) - I\,| < \varepsilon,$$

with accomplishes the proof. ◇

**3.8. <u>Corollary.</u>** The surface integral of the second type does not depend on parameterization (as long as we present the orientation).

<u>Proof.</u> According to corollary 4 in § VIII.2, the surface integral of first type is independent of parameterization. Restricting to positive diffeomorphisms of the orientated surface $S$, $\vec{n}$ is also an invariant of the surface, hence the integrals in 3.7(*ii*) from above do not depend on parameterization. ◇

**3.9. <u>Proposition.</u>** The surface integral of the second order has the properties:

(i) $\iint\limits_S (\alpha\vec{V} + \beta\vec{W})\cdot\vec{n}\,dS = \alpha\iint\limits_S \vec{V}\cdot\vec{n}\;dS + \beta\iint\limits_S \vec{W}\cdot\vec{n}\,dS$ (*linearity*)

(ii) $\iint\limits_{S_1\cup S_2} \vec{V}\cdot\vec{n}\,dS = \iint\limits_{S_1} \vec{V}\cdot\vec{n}\,dS + \iint\limits_{S_2} \vec{V}\cdot\vec{n}\,dS$ (*additivity*), whenever $S_1$ and $S_2$ have at most frontier common points

(iii) $\iint\limits_{S^-} \vec{V}\cdot\vec{n}\,dS = -\iint\limits_{S} \vec{V}\cdot\vec{n}\,dS$ (*orientation*) where $S^-$ is the contrary orientated surface (of normal $-\vec{n}$).

<u>Proof.</u> (i) and (ii) are consequences of proposition 7, §2. Property (iii) simply follows from $\vec{V}\,(-\vec{n}) = -\vec{V}\cdot\vec{n}$, and (*i*) . ◇

**3.10. <u>Remark.</u>** Using parameterizations, we may reduce formula (ii) from theorem 3.7 to several double integrals as follows:

$$\iint_S \vec{V}\cdot\vec{n}\,dS = \iint_D (V_1A+V_2B+V_3C)dudv = \iint_D \vec{V}\cdot(\vec{r}_u\times\vec{r}_v)dudv =$$
$$= \iint_{\mathrm{Pr}_{xy}(S)} (-V_1p - V_2q + V_3)dxdy\,.$$

Obviously, these formulas correspond to different forms of the expression of the normal $\vec{n}$, and that of the elementary area *dS*.

**3.11. <u>Example</u>.** Let us evaluate the integral

$$I = \iint_S xzdydz + yzdzdx + (x^2+y^2)dxdy\,,$$

where *S* denotes the upwards orientated surface of equation $z = x^2 + y^2$, restricted to the condition $z \le 1$.

We may start by writing the normal vector, for example in the form

$$\vec{n} = \frac{1}{\sqrt{1+4x^2+4y^2}}(-2x\vec{i} - 2y\vec{j} + \vec{k})\,.$$

Consequently, we may reduce the problem to a surface integral of the first type, i.e.

$$I = \iint_S \frac{1}{\sqrt{1+4x^2+4y^2}}\left[-2x^2z - 2y^2z + (x^2+y^2)\right]dS\,.$$

Further on, this integral reduces to a double one by replacing *dS*, e.g.

$$I = \iint_D (1-2x^2-2y^2)(x^2+y^2)dxdy\,,$$

where $D = \{(x,y) : x^2+y^2 \le 1\}$. Using polar coordinates, we easily obtain the result $I = -\dfrac{\pi}{6}$ .

## PROBLEMS § VIII.3

**1.** Evaluate $I = \iint\limits_S yzdydz + xzdzdx + xydxdy$, when $S$ is the external side of the tetrahedron bounded by the planes of equations $x = 0$, $y = 0$, $z = 0$, and $x + y + z = a > 0$.

Hint. The integral on the side $x = 0$ reduces to $\int\limits_0^a \left[ y \int\limits_0^{a-y} zdz \right] dy$, etc. For the side $S_a$ of equation $x + y + z = a$ we have $\vec{n} = \frac{1}{\sqrt{3}}(\vec{i} + \vec{j} + \vec{k})$, hence the integral can be expressed as an integral of the first type

$$\frac{1}{\sqrt{3}} \iint\limits_{S_a} (yz + xz + xy)dS .$$

**2.** Find the flux of the vector function $\vec{V}$ $(x^2, y^2, z^2)$ through the sphere $(x - a)^2 + (y - b)^2 + (z - c)^2 = R^2$.

Hint. $\Phi = \iint\limits_S V_1 dydz + V_2 dzdx + V_3 dxdy$. In particular,

$$\vec{n} = \frac{1}{R}(x - a, y - b, z - c),$$

hence $\Phi$ reduces to a surface integral of the first type

$$\Phi = \frac{1}{R} \iint\limits_S [x^2(x-a) + y^2(y-b) + z^2(z-c)]dS .$$

Using spherical coordinates is advisable, since $dS = R^2 sin\theta\, d\theta d\varphi$, and

$$\Phi = \frac{8}{3}\pi R^3(a + b + c) .$$

**3.** Evaluate $I = \iint\limits_S zdxdy$, where $S$ is the external side of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, and interpret the result.

Hint. Using the parametric equations of the ellipsoid

$$x = a \sin\theta \cos\varphi, \quad y = b \sin\theta \sin\varphi, \quad z = c \cos\theta,$$

we obtain

$$\vec{n} = (b\, c \sin^2\theta \cos\varphi,\ a\ c \sin^2\theta \sin\varphi,\ a\, b \sin\theta \cos\theta),$$

hence

$$I = \iint_D c\cos\theta\, ab\sin\theta\cos\theta d\varphi d\theta = 2\pi abc\int_0^{\pi}\cos^2\theta\sin\theta d\theta = \frac{4\pi}{3}abc.$$

**4.** Evaluate $I = \iint_S \frac{dydz}{x} + \frac{dzdx}{y} + \frac{dxdy}{z}$, where $S$ is the exterior side of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Hint. $I$ is apparently improper since $x$, $y$, $z$ can be zero on $S$, but if we introduce the parametric equation of the ellipsoid (as above), it becomes a definite double integral; $I = 4\pi\left(\frac{bc}{a} + \frac{ac}{b} + \frac{ab}{c}\right)$.

**5.** Let $S$ be a closed regular surface, which bounds a measurable domain $\Omega$, such that each parallel to $ox$, $oy$, $oz$ axis meets $S$ at most two times. Show that the volume of $\Omega$ is given by

$$v(\Omega) = \frac{1}{3}\iint_S xdydz + ydzdx + zdxdy.$$

Hint. $v(\Omega) = \iint_S zdxdy$, since $S = S_1 \cup S_2$, where

$$S_1 = \{(x, y, z) : z = f_1(x, y), (x, y) \in D\}$$
$$S_2 = \{(x, y, z) : z = f_2(x, y), (x, y) \in D\},$$

and $D = Pr_{xy}(S)$.

Supposing $f_1 > f_2$, and taking into consideration the orientation,

$$\iint_S zdxdy = \iint_D f_1 dxdy - \iint_D f_2 dxdy.$$

Similarly, we treat the other projections (see also problem 3).

§ VIII.4. INTEGRAL FORMULAS

Our purpose in this section is to establish relations between line, surface, and multiple integrals in $\mathbb{R}^3$. A similar relation between line and double integral in $\mathbb{R}^2$ we already have discussed in theorem 21, §2, chapter VII, where we have proved the Green integral formula.

$$\int_{\gamma} Pdx + Qdy = \iint_{D} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy .$$

A characteristic of these formulas consists in some specific restrictions on the considered domain and its frontier, which will be included in the following definition:

**4.1. <u>Definition.</u>** We say that the domain $\mathscr{D} \subset \mathbb{R}^3$ is *regular* iff it satisfies the conditions:

(i) $\mathscr{D}$ is a measurable compact domain (*m.c.d.*)

(ii) $\mathscr{D}$ is a finite union of simple sub-domains relative to all axes (i.e. any line parallel to *ox, oy* or *oz* meets the frontier $S$ of $\mathscr{D}$ at most two times), without common interior points.

(iii) $S = Fr(\mathscr{D})$ is a regular, closed and orientated surface.

For regular domains the triple integral may be expressed by a surface integral of the second type as follows:

**4.2. <u>Theorem.</u>** (Gauss-Ostrogradski formula) If $\mathscr{D} \subset \mathbb{R}^3$ is a regular domain of frontier $S$, and $\vec{V} \in C_{\mathbb{R}}^1(\mathscr{D})$ is a vector function of components $V_1$, $V_2$, $V_3$, then

$$\iiint_{\mathscr{D}} \left( \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z} \right) dxdydz = \iint_{S} V_1 dydz + V_2 dzdx + V_3 dxdy .$$

<u>Proof.</u> If $\mathscr{D} = \mathscr{D}_1 \cup \ldots \cup \mathscr{D}_n$, as above, it is sufficient to prove the formula for $\mathscr{D}_k$, $k = 1, \ldots, n$. More exactly, we can show only that

$$\iiint_{\mathscr{D}_k} \frac{\partial V_3}{\partial z} dxdydz = \iint_{Fr(\mathscr{D}_k)} V_3 dxdy ,$$

because adding the similar formulas for $V_1$ and $V_2$ on all $\mathscr{D}_k$, $k = 1, \ldots, n$, we obtain the claimed formula.

In fact, since $\mathscr{D}_k$ is simple relative to *oz* axes, there exist $f_k, g_k : Pr_{xy}(\mathscr{D}_k) \to \mathbb{R}$ such that

$$\mathscr{D}_k = \{(x, y, z) \in \mathbb{R}^3 : f_k(x, y) \le z \le g_k(x, y), (x, y) \in Pr_{xy}(\mathscr{D}_k)\}.$$

By iterating the triple integral on $\mathscr{D}_k$ we obtain

$$\iiint\limits_{\mathcal{D}_k} \frac{\partial V_3}{\partial z} dxdydz = \iint\limits_{\mathrm{Pr}_{xy}(\mathcal{D}_k)} \left[ \int\limits_{f_k(x,y)}^{g_k(x,y)} \frac{\partial V_3}{\partial z} dz \right] dxdy =$$

$$= \iint\limits_{\mathrm{Pr}_{xy}(\mathcal{D}_k)} [V_3(x, y, g_k(x, y)) - V_3(x, y, f_k(x, y))] dxdy =$$

$$= \iint\limits_{S_{k,2}} V_3 dxdy - \iint\limits_{S_{k,2}^-} V_3 dxdy,$$

where

$$S_{k,1} = \{(x, y, z) \in \mathbb{R}^3 : z = f_k(x, y), (x, y) \in Pr_{xy}(\mathcal{D}_k\}$$

and

$$S_{k,2} = \{(x, y, z) \in \mathbb{R}^3 : z = g_k(x, y), (x, y) \in Pr_{xy}(\mathcal{D}_k)\}.$$

The sign "–" at $S_{k,2}^-$ shows that the positive sense of the normal is opposite to the usual one (in accordance with the sense of the *oz* axes). Using the orientation of the surface integral of the second type relative to the normal, we may remark that $S_k = S_{k,1} \cup S_{k,2}$ is the frontier surface of $\mathcal{D}_k$, hence we have

$$\iiint\limits_{\mathcal{D}_k} \frac{\partial V_3}{\partial z} dxdydz = \iint\limits_{S_k} V_3 dxdy.$$

Similarly we treat the other integrals. ◇

**4.3. <u>Remark.</u>** (i) Expressing the surface integral in Gauss-Ostrogradski formula by a surface integral of the first type we obtain

$$\iiint\limits_{\mathcal{D}} \left( \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z} \right) dxdydz = \iint\limits_{S} \vec{V} \cdot \vec{n}\, dS \ ,$$

where the last integral represents the flux of $\vec{V}$ through $S$. The triple integral can also be simplified if we define the *divergence* of $\vec{V}$ as

$$div\ \vec{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}.$$

In this case the Gauss-Ostrogradski formula takes the form

$$\iiint\limits_{\mathcal{D}} div\ \vec{V}\, dxdydz = \iint\limits_{S} \vec{V} \cdot \vec{n}\, dS,$$

also called *the flux-divergence* formula. It is very useful in field theory by its remarkable consequences (see the next chapter).

(ii) The other important integral formula relates line and surface integrals involving the notion of rotation. Therefore we recall (see definition 11, §3, chapter VI) that the *rotation* of $\vec{V} = (V_1, V_2, V_3) \in C^1_{\mathbb{R}^3}(\mathcal{D})$ is defined by

$$rot\ \vec{V} = \left( \frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z} \right) \vec{i} + \left( \frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x} \right) \vec{j} + \left( \frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y} \right) \vec{k} =$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}.$$

There are also necessary some regularity conditions for the surface.

**4.4. <u>Definition.</u>** We say that surface $S$ is *explicit* relative to $z$ iff there exists an open set $O \subseteq \mathbb{R}^2$, a function $f \in C^2_{\mathbb{R}}(O)$, and a *m.c.d.* $D \subset O$ such that

$$S = \{(x, y, z) \in \mathbb{R}^3 : z = f(x, y), (x, y) \in D\}.$$

Similarly, we define the explicit surface relative to $x$ or $y$. If $S$ is explicit relative to $x$, $y$, or $z$, then we simply say that $S$ is *explicit*. Surface $S$ is called <u>elementary</u> iff it consists of a finite number of regular and explicit surfaces.

**4.5. <u>Remark.</u>** Each explicit surface is orientated according to the convention in § VIII.3. In fact, if $S$ is a regular surface explicit relative to $z$, then the curve $\gamma = \mathrm{Fr}D$ has a natural positive sense, namely the counter-clockwise one, which induces the positive sense on

$$\Gamma = \{(x, y, z) \in \mathbb{R}^3 : z = f(x, y), (x, y) \in \gamma\}.$$

Usually, $\Gamma$ is called the *orientated border* of $S$. This orientation is compatible with that of $S$ in the sense of the right-hand screw rule.

When the elementary surface $S$ is decomposed in regular and explicit sub-surfaces, by convenience we consider that these sub-surfaces have only border points in common. More exactly, each part of the border of a sub-surface can belong to at most two sub-surfaces, case in which it is traced in both opposite senses. The union of all parts of the borders which belong to a single sub-surface form the *border* of $S$, denoted $\Gamma = Bd(S)$.

**4.6. <u>Theorem.</u>** (Stokes formula) Let $\vec{V} \in C^1_{\mathbb{R}^3}(\mathscr{D})$ be a vector function of components $V_1$, $V_2$, $V_3$ on the domain $\mathscr{D} \subseteq \mathbb{R}^3$. If $S \subset \mathscr{D}$ is an elementary surface of border $\Gamma$, then

$$\int_\Gamma \vec{V}\, d\vec{r} = \iint_S (rot\vec{V}) \cdot \vec{n}\, dS.$$

<u>Proof.</u> It is sufficient to prove the formula for a single sub-surface of $S$ which is regular and explicit relative to $z$ (for example), because finally we can add such relations to obtain the claimed one. In other terms, we will prove the formula supposing that $S$ reduces to a single regular surface, which is explicit relative to $z$.

Let $\varphi : [a, b] \to \mathbb{R}^3$ be a parameterization of $\Gamma = Bd(S)$. If we explicit $\varphi(t) = (x(t), y(t), z(t))$ for all $t \in [a, b]$, then

$$\int_\Gamma \vec{V}\, d\vec{r} = \int_\Gamma V_1 dx + V_2 dy + V_3 dz =$$

$$= \int_a^b [(V_1 \circ \varphi)(t)x'(t) + (V_2 \circ \varphi)(t)y'(t) + (V_3 \circ \varphi)(t)z'(t)]\, dt.$$

Because $\Gamma = Bd(S)$ is a part of $S$, we have $z(t) = f(x(t), y(t))$ on $\Gamma$, hence

$$z'(t) = \frac{\partial f}{\partial x}(x(t), y(t))x'(t) + \frac{\partial f}{\partial y}(x(t), y(t))y'(t).$$

Consequently,

$$\int_\Gamma \vec{V}\, d\,\vec{r} = \int_a^b [(V_1 + V_3\frac{\partial f}{\partial x})\circ\varphi](t)x'(t) + [(V_2 + V_3\frac{\partial f}{\partial y})\circ\varphi](t)y'(t)dt =$$

$$= \int_\gamma [V_1(x,y,f(x,y)) + V_3(x,y,f(x,y))\frac{\partial f}{\partial x}(x,y)]dx +$$

$$+ [V_2(x,y,f(x,y)) + V_3(x,y,f(x,y))\frac{\partial f}{\partial y}(x,y)]dy.$$

Using the Green formula for $\gamma$ and $D$ in $\mathbb{R}^2$, we obtain

$$\int_\Gamma \vec{V}\, d\,\vec{r} = \iint_D \left[\frac{\partial}{\partial x}(V_2 + V_3\frac{\partial f}{\partial y}) - \frac{\partial}{\partial y}(V_1 + V_3\frac{\partial f}{\partial x})\right]dxdy$$

The problem reduces to evaluating the square bracket under this double integral. In fact, since $f \in C_{\mathbb{R}}^2(D)$, its mixed partial derivatives of the second order are equal, hence

$$\frac{\partial}{\partial x}(V_2 + V_3\frac{\partial f}{\partial y}) - \frac{\partial}{\partial y}(V_1 + V_3\frac{\partial f}{\partial x}) =$$

$$= \frac{\partial V_2}{\partial x} + \frac{\partial V_2}{\partial z}\frac{\partial f}{\partial x} + \left(\frac{\partial V_3}{\partial x} + \frac{\partial V_3}{\partial z}\frac{\partial f}{\partial x}\right)\frac{\partial f}{\partial y} + V_3\frac{\partial^2 f}{\partial x\partial y} -$$

$$-\frac{\partial V_1}{\partial y} - \frac{\partial V_1}{\partial z}\frac{\partial f}{\partial y} - \left(\frac{\partial V_3}{\partial y} + \frac{\partial V_3}{\partial z}\frac{\partial f}{\partial y}\right)\frac{\partial f}{\partial x} - V_3\frac{\partial^2 f}{\partial x\partial y} =$$

$$= -\frac{\partial f}{\partial x}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) - \frac{\partial f}{\partial y}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right) =$$

$$= (rot\vec{V})\cdot\vec{n}\cdot\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1},$$

where $\vec{n} = \left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1\right]^{-1/2}\left(-\frac{\partial f}{\partial x}\vec{i} - \frac{\partial f}{\partial y}\vec{j} + \vec{k}\right)$ is the unit normal to $S$ ($\|\vec{n}\| = 1$). Finally,

$$\int_\Gamma \vec{V}\, d\,\vec{r} = \iint_D (rot\vec{V})\cdot\vec{n}\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\; dxdy = \iint_S (rot\vec{V})\cdot\vec{n}\, dS,$$

which proves the Stokes formula. ◇

**4.7. <u>Remarks.</u>** (i) The Green formula (which has been used in the proof) is a particular form of the Stokes formula. In fact, if $V_3 = 0$, and $\Gamma = \gamma$ is a plane curve bordering the domain $S = D \subset \mathbb{R}^2$, then $\vec{n} = (0,0,1)$, hence we have $(rot\ \vec{V})\vec{n} = \frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}$, while

$$\int_{\Gamma} \vec{V}\, d\,\vec{r} = \int_{\gamma} V_1 dx + V_2 dy.$$

(ii)The line integral $\int_{\Gamma} \vec{V}\, d\,\vec{r}$ is also called the *curl* or *circulation* of $\vec{V}$ on $\Gamma$. Using this term the Stokes formula says that: "the flux of the rotation of $\vec{V}$ through $S$ is equal to the curl of $\vec{V}$ along the border $\Gamma$ of $S$" .

**4.8. <u>Corollary.</u>** Under the conditions of theorem 4.6, if $S_1$ and $S_2$ are elementary surfaces having the same border $\Gamma$, then the fluxes of *rot* $\vec{V}$ through $S_1$ and $S_2$ are equal.

<u>Proof.</u> According to the Stokes formula both fluxes are equal to the curl of $\vec{V}$ on $\Gamma$. Obviously, the orientation of $S_1$ and $S_2$ are supposed to be compatible to the positive sense on $\Gamma$. ◇

**4.9. <u>Remark.</u>** Using Stokes formula we can improve theorem 12, §3, chapter VI, in the sense that the condition for the domain to be stationary can be removed from the hypothesis. In fact, if the field $\vec{V}$ is conservative, i.e. *rot* $\vec{V} = 0$, then the curl on any closed curve is null, hence the line integral of the second type does not depend on the curve, but only on its endpoints. In other terms, each *irrotational* field is *non-circulatory* (or *circulation free*).

We mention that besides their theoretical importance (obviously in field theory), the above formulas are frequently useful in order to evaluate surface and line integrals.

## PROBLEMS §VIII.4

**1.** Evaluate $I = \iint_S yz\,dydz + zx\,dzdx + xy\,dxdy$, where $S$ is the boundary of a regular domain $\mathcal{D} \subset \mathbb{R}^3$. Generalization.

Hint. $\vec{V}$ $(x, y, z) = (yz, zx, xy)$, hence $div\ \vec{V} = 0$, and $I = 0$ according to the Gauss-Ostrogradski formula. More generally, we obtain a null integral if $\vec{V}$ $(x, y, z) = (f(y, z), g(x, z), h(x, y))$.

**2.** Evaluate $I = \iint_S x\,dydz + y\,dzdx + z\,dxdy$, where $S$ is the external surface of a sphere of radius $r$ (and arbitrary center).

Hint. In the Gauss-Ostrogradski formula $div\ \vec{V} = 3$, and $\iiint_{\mathcal{D}} dxdydz$ is the volume of the sphere.

**3.** Find $I = \iint_S x^2dydz + y^2dxdz + z^2dxdy$ if $S$ is the external total surface of the cone $0 \leq z = h\sqrt{x^2 + y^2} \leq h$, where $h > 0$.

Hint. The Gauss-Ostrogradski formula reduces $I$ to a triple integral; the result is $I = \dfrac{\pi h^2}{2}$.

**4.** Show that if $\vec{V}$ derives from a harmonic potential in the regular domain $\mathcal{D}$, then the flux $\iint_S \vec{V} \cdot \vec{n}\,dS = 0$, where $S = \mathrm{Fr}\,(\mathcal{D})$.

Hint. By hypothesis, $\vec{V} = \left(\dfrac{\partial U}{\partial x}, \dfrac{\partial U}{\partial y}, \dfrac{\partial U}{\partial z}\right)$, hence $div\ \vec{V} = \Delta U = 0$ because $U$ is harmonic. Use the Gauss-Ostrogradski formula.

**5.** Prove that if $S$ is a closed surface, which bounds a regular domain, and $\vec{l}$ is a fixed direction, then $I = \iint_S \cos(\vec{n}, \vec{l})dS = 0$, where $\vec{n}$ is the outer normal to $S$.

Hint. Consider $\vec{n} = (\cos\alpha, \cos\beta, \cos\gamma)$ and $\vec{l} = (\cos\alpha_0, \cos\beta_0, \cos\gamma_0)$, such that $I = \iint_S \cos\alpha_0 dydz + \cos\beta_0 dzdx + \cos\gamma_0 dxdy$. On the other hand, $\cos(\vec{n}, \vec{l}) = \langle \vec{n}, \vec{l} \rangle$, as in the Gauss-Ostrogradski formula.

**6.** Evaluate $I = \int\limits_{\Gamma} y^2dx + z^2dy + x^2dz$, where $\Gamma$ is the contour of the triangle of vertices $A(a, 0, 0)$, $B(0, a, 0)$, $C(0, 0, a)$.

Hint. $\vec{V} = (y^2, z^2, x^2)$ has $rot\ \vec{V} = (-2z, -2x, -2y)$, hence using Stokes formula, $I = -\frac{2}{\sqrt{3}}\iint\limits_{S}(x + y + z)dS$, where $S$ is the surface of the triangle $ABC$. Since $x + y + z = a$ on $S$, and $\iint\limits_{S} dS$ is the area of $\Delta\, ABC$, we obtain $I = -a^3$.

**7.** Applying Stokes formula, find $I = \int\limits_{\Gamma}(y - z)dx + (z - x)dy + (x - y)dz$, where $\Gamma$ is the ellipse of equations $x^2 + y^2 = 1$, $x + z = 1$. Verify the result by direct calculation.

Hint. $\vec{V} = (y - z, z - x, x - y)$ has $rot\ \vec{V} = -2(\vec{i} + \vec{j} + \vec{k})$, and the plane of the ellipse has $\vec{n} = \frac{1}{\sqrt{2}}(1, 0, 1)$. $I = -4\pi$. A parameterization of $\Gamma$ is $x = cos\ t$, $y = sin\ t$, $z = 1 - cos\ t$, $t \in [0, 2\pi]$.

**8.** Evaluate the line integral $I = \int\limits_{\Gamma} xdx + (x + y)dy + (x + y + z)dz$, where $\Gamma$ has the parameterization $x = a\ cos\ t$, $y = a\ sin\ t$, $z = a(cos\ t + sin\ t)$, $t \in [0, 2\pi]$, using Stokes' formula, and directly.

Hint. $rot\ \vec{V} = \vec{i} - \vec{j} + \vec{k}$, and $\Gamma$ is an ellipse on the plane $z = x + y$.

**9.** Find the curl of $\vec{V} = \frac{2x}{x^2 + y^2}\vec{i} + \frac{2y}{x^2 + y^2}\vec{j} + 2z\vec{k}$ along the circle of equations $x^2 + y^2 = 1$, $z = 1$ traced once in the positive sense relative to $oz$–axis.

Hint. $rot\ \vec{V} = 0$, hence apply the Stokes' formula.

# CHAPTER IX. ELEMENTS OF FIELD THEORY

In essence, all the important notations of the field theory were already introduced and studied in the previous chapter for both scalar and vector fields. Therefore this chapter will be a synthesis on the differential and integral calculus, expressed in a more intuitive language, specific to applications. For these practical purposes, in § IX.3, we will put forward the most significant types of particular fields.

## § IX.1. DIFFERENTIAL OPERATORS

For the beginning, we have to clarify the notion of field, which so far was reduced to a scalar function $\varphi : \mathscr{D} \to \mathbb{R}$, when we were speaking about scalar fields, or to a vector function $\vec{V} : \mathscr{D} \to \mathbb{R}^3$, in the case of a vector field. Usually, $\mathscr{D}$ is a domain in $\mathbb{R}^3$, but a similar topic is valid when $\mathscr{D} \subseteq \mathbb{R}^2$. Some problems arise when operating with $\varphi$ and $\vec{V}$, since the values of $\varphi$ are considered as belonging to the field of real numbers, over which the vector space $\mathbb{R}^3$ is defined, and the space $\mathbb{R}^3$ of the values of $\vec{V}$ is identified with the initial vector space $\mathbb{R}^3$, which contains $\mathscr{D}$. In other terms, as long as $\mathbb{R}^3$ is a set of pints $(x, y, z)$, or position vectors

$$\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}\,,$$

the definition of $\vec{V}\,(x, y, z)$ in the same space, as in Fig. IX.1.1, makes no rigorous meaning in spite of its practical use (e.g. the work of a force, the flux, etc.). This situation is clarified by considering the notion of "tangent" space:

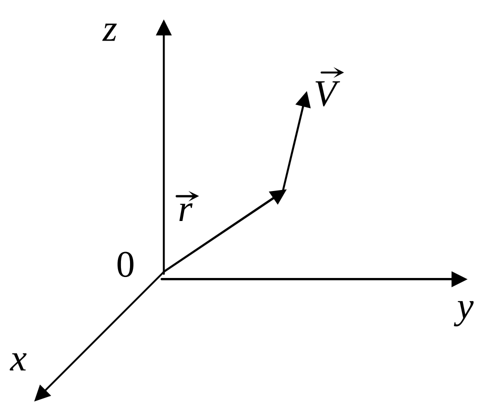



Fig. IX.1.1.

**1.1. <u>Definition.</u>** Let $A = (x_0, y_0, z_0)$ be a fixed point in $\mathbb{R}^3$. For any other $B \in \mathbb{R}^3$, the pair $(A, B)$ is called *tangent vector* at $A$ to $\mathbb{R}^3$. The set of all tangent vectors at $A$ is called *tangent space* at $A$, and is denoted as $T_A$.

The point $A$ is called *origin* (or *application point*), and $B$ is called *vertex* of the tangent vector $(A, B)$. The number $\|B - A\|$ is the *length*, and $B - A$ is the *vector part* of $(A, B)$. It is easy to organize $T_A$ as a linear space:

**1.2. <u>Proposition.</u>** $T_A$ endowed with the operations $\oplus$ and $\otimes$ defined by

$$(A, B) \oplus (A, C) = (A, B + C - A)$$
$$\lambda \otimes (A, B) = (A, A + \lambda(B - A))$$

is a linear space isometric to $\mathbb{R}^3$.

The proof is routine.

**1.3. <u>Remark.</u>** The tangent space reproduces the geometry of $\mathbb{R}^3$ at $A$ since we can define the *scalar product* of two tangent vectors using the scalar product in $\mathbb{R}^3$ of their vector parts, i.e.

$$<(A, B), (A, C)> = (B - A)(C - A).$$

Using this notion we can introduce the notions of norm, distance, angle orthogonality, etc., and we can see that the correspondence

$$T_A \ni (A, B) \leftrightarrow B - A \in \mathbb{R}^3$$

is an isometric isomorphism.

The tangent vectors

$\vec{i}_A = (A, (1, 0, 0) + A)$, $\vec{j}_A = (A, (0, 1, 0) + A)$, and $\vec{k}_A = (A, (0, 0, 1) + A)$

represent the *canonical basis* of $T_A$. Using the *components* of the tangent vectors we can also construct the *vector product*, the *mixed product*, etc.

Between tangent vectors of different origins we have the relation of *parallelism* defined by

$$(A_1, B_1) \,||\, (A_2, B_2) \Leftrightarrow 0, B_1 - A_1 \text{ and } B_2 - A_2 \text{ are collinear.}$$

Now we can formulate the correct notion of *vector fields*, which is also applicable to general (non-flat) manifolds:

**1.4. <u>Definition.</u>** The set $\mathscr{T} = \bigcup_{A \in \mathbb{R}^3} T_A$ is called *tangent bundle* of $\mathbb{R}^3$. A *vector field* in the domain $\mathscr{D} \subseteq \mathbb{R}^3$ is a function $\vec{V} : \mathscr{D} \to \mathscr{T}$ for which $\vec{V}(A) = \vec{V}_A \in T_A$ for all $A \in \mathscr{D}$.

If $\vec{V}$ and $\vec{W}$ are vector fields on $\mathscr{D}$, their *sum* is defined by

$$(\vec{V} + \vec{W})(A) = \vec{V}_A \oplus \vec{W}_A$$

at any $A \in \mathscr{D}$. Similarly, if $\vec{V}$ is a vector field on $\mathscr{D}$ and $f : \mathscr{D} \to \mathbb{R}$ is a scalar field, their *product* is defined by

$$(f\vec{V})(A) = f(A) \otimes \vec{V}_A$$

at any $A \in \mathscr{D}$.

Similarly, we can define the *scalar product*, the *vector product*, etc., of vector fields using, at each $A \in \mathscr{D}$, the corresponding operations in $T_A$, i.e. by "local" constructions.

In order to justify the previous use of the term "vector field" for functions $\vec{V} : \mathscr{D} \to \mathbb{R}^3$, where $\mathscr{D} \subseteq \mathbb{R}^3$, we mention that in the case of $\mathbb{R}^3$ (which is a linear, "flat" manifold) we have:

**1.5. <u>Proposition.</u>** If $\mathscr{V}$ is the space of all vector fields on $\mathscr{D} \subseteq \mathbb{R}^3$ and $\mathscr{F}$ is the set of all vector functions on $\mathscr{D}$, then $\mathscr{V}$ and $\mathscr{F}$ are isometrically isomorphic.

<u>Proof.</u> Each vector function $\vec{F} : \mathscr{D} \to \mathbb{R}^3$ is defined by three components, i.e. $\vec{F} = (f_1, f_2, f_3)$, which are scalar functions on $\mathscr{D}$. It is easy to see that each vector field is also defined by three components, i.e.

$$\vec{V} = V_1 \vec{I} + V_2 \vec{J} + V_3 \vec{K},$$

where $V_1, V_2, V_3 : \mathscr{D} \to \mathbb{R}$. In fact, if $\vec{I}, \vec{J}, \vec{K}$ represent the *fundamental* fields, defined at any $A \in \mathscr{D}$ by

$$\vec{I}(A) = \vec{i}_A = (A, (1, 0, 0) + A)$$
$$\vec{J}(A) = \vec{j}_A = (A, (0, 1, 0) + A)$$
$$\vec{K}(A) = \vec{k}_A = (A, (0, 0, 1) + A)$$

then $V_1 = <\vec{V}, \vec{I}>$, $V_2 = <\vec{V}, \vec{J}>$, $V_3 = <\vec{V}, \vec{K}>$.

The claimed isomorphism is obtained by identifying the corresponding components $V_1, V_2, V_3$ and $f_1, f_2, f_3$. ◇

**1.6. <u>Remarks</u>.** (i) The study of the scalar and vector fields is realized by three differential operators: gradient, divergence and rotation, which can be unitarily treated using the following Hamilton's "nabla" (or "del") operator (The Greek **νάβλα** is the name of an ancient musical instrument of triangular shape):

$$\nabla = \frac{\partial}{\partial x} \vec{I} + \frac{\partial}{\partial y} \vec{J} + \frac{\partial}{\partial z} \vec{K}.$$

The constant fields $\vec{I}, \vec{J}, \vec{K}$ are mentioned here in order to emphasize the local character of nabla, but according to the above proposition we can simply note

$$\nabla = \frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j} + \frac{\partial}{\partial z} \vec{k}.$$

(ii) For practical uses the symbol $\nabla$ can take two meanings, namely that of a vector, and that of an operator. As an operator, which contains the partial derivatives, it manifests also two characteristics, namely:

- linear operator relative to the algebraic operations;
- differential operator acting on the components of the field.

These properties of $\nabla$ determine the rules of operating with it and also the significance of its action.

(iii) If $U : \mathscr{D} \to \mathbb{R}$ is a scalar field, then

$$\nabla U = \frac{\partial U}{\partial x}\vec{i} + \frac{\partial U}{\partial y}\vec{j} + \frac{\partial U}{\partial z}\vec{k} = grad\ U.$$

If $\vec{V} : \mathscr{D} \to \mathscr{T}$ is a vector field of components $V_1$, $V_2$, $V_3$, then

$$\nabla \cdot \vec{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z} = div\ \vec{V}, \text{ and}$$

$$\nabla \times \vec{V} = \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right)\vec{i} + \left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right)\vec{j} + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)\vec{k} = rot\ \vec{V}.$$

We mention that $\nabla$ occurs in the notion of derivative of a scalar field $U$ along the unit vector $\vec{l}$, i.e. $\frac{\partial}{\partial \vec{l}} = \vec{l} \cdot \nabla$, in the sense that

$$(\vec{l} \cdot \nabla)U = \vec{l} \cdot (\nabla U) = \vec{l} \cdot grad\ U = \frac{\partial U}{\partial \vec{l}}.$$

The derivative of a vector field $\vec{V}$ in direction $\vec{l}$, which is defined by

$$\frac{\partial \vec{V}}{\partial \vec{l}}(\vec{x}) = \lim_{t \to 0} \frac{\vec{V}(\vec{x} + t\vec{l}) - \vec{V}(\vec{x})}{t},$$

may be similarly expressed as:

$$(\vec{l} \cdot \nabla)\vec{V} = \left(l_x \frac{\partial}{\partial x} + l_y \frac{\partial}{\partial y} + l_z \frac{\partial}{\partial z}\right)\vec{V} = \frac{\partial V_1}{\partial \vec{l}}\vec{i} + \frac{\partial V_2}{\partial \vec{l}}\vec{j} + \frac{\partial V_3}{\partial \vec{l}}\vec{k}.$$

In such formulas $\vec{l} \cdot \nabla$ acts as a *scalar differential operator*.

The Laplace second order differential operator on scalar fields

$$\Delta U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}$$

is frequently considered as $\Delta = \nabla^2$, in the sense that

$$\Delta U = (\nabla \cdot \nabla)U = \nabla \cdot (\nabla U) = div\ (grad\ U).$$

The vectorial behavior of $\nabla$ is visible in the following:

**1.7. <u>Proposition.</u>** For any scalar field $U$ and vector field $\vec{V}$ we have:

(i) $\nabla \times (\nabla U) = 0_{\mathbb{R}^3}$;

(ii) $\nabla \cdot (\nabla \times \vec{V}) = 0$;

(iii) $\nabla \times (\nabla \times \vec{V}) = \nabla(\nabla \cdot \vec{V}) - (\nabla \cdot \nabla)\vec{V}$.

<u>Proof.</u> (i) The vector product of collinear vectors is null; in this case it means that $rot\ (grad\ U) = \vec{0}$.

(ii) The mixed product, in which two of the vectors are collinear, is null. In other words $div\ (rot\ \vec{V}) = 0$.

(iii) $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$ is generally valid for the double vector product of three vectors, hence also for $\vec{a} = \vec{b} = \nabla$ and $\vec{c} = \vec{V}$. In particular, this formula shows that

$$rot\,(rot\,\vec{V}) = grad\,(div\,\vec{V}) - \Delta\vec{V},$$

where $\Delta\vec{V} = \Delta V_1 \vec{i} + \Delta V_2 \vec{j} + \Delta V_3 \bar{k} = \dfrac{\partial^2 \vec{V}}{\partial x^2} + \dfrac{\partial^2 \vec{V}}{\partial y^2} + \dfrac{\partial^2 \vec{V}}{\partial z^2}$.

We remember that the starting formula follows from

$$\vec{a} \times (\vec{b} \times \vec{c}) = \lambda\vec{b} + \mu\vec{c}.$$

We multiply by $\vec{a}$ to obtain $\lambda(\vec{a}\cdot\vec{b}) + \mu(\vec{a}\cdot\vec{c}) = 0$, hence $\lambda = k(\vec{a}\cdot\vec{c})$ and $\mu = -k(\vec{a}\cdot\vec{b})$. If we take $\|\vec{a}\| = \|\vec{b}\| = \|\vec{c}\| = 1$, and $\vec{a}=\vec{b} \perp \vec{c}$, then we obtain $k=1$.

The linear character of $\nabla$ is essential in properties as:

**1.8. <u>Proposition.</u>** Let $T$, $U$ be scalar fields on $\mathscr{D} \subseteq \mathbb{R}^3$, $\vec{V}$, $\vec{W}$ be vector fields on $\mathscr{D}$, and $\lambda \in \mathbb{R}$. Then the following formulas hold:

(i) $\nabla(U + T) = \nabla U + \nabla T$; $\nabla(\lambda U) = \lambda\nabla U$

(ii) $\nabla\cdot(\vec{V}+\vec{W}) = \nabla\cdot\vec{V}+\nabla\cdot\vec{W}$; $\nabla\cdot(\lambda\vec{V}) = \lambda\nabla\cdot\vec{V}$

(iii) $\nabla \times (\vec{V}+\vec{W}) = \nabla \times \vec{V} + \nabla \times \vec{W}$; $\nabla \times (\lambda\vec{V}) = \lambda\nabla \times \vec{V}$.

<u>Proof.</u> These formulas express the linearity of ***grad*, *div*** and ***rot***. ◇

The property of $\nabla$ of being a differential operator is especially visible whenever it acts on a product.

**1.9. <u>Proposition.</u>** If $U$, $T$ are scalar fields, and $\vec{V}$, $\vec{W}$ are vector fields, then:

(i) $\nabla U = 0$ if and only if $U$ = constant

(ii) $\nabla\cdot\vec{V} = 0$ if $\vec{V}$ is constant

(iii) $\nabla \times \vec{V} = 0$ if $\vec{V}$ is constant

(iv) $\nabla(UT) = T\,\nabla U + U\nabla T$

(v) $\nabla\cdot(U\vec{V}) = \vec{V}\cdot(\nabla U) + U(\nabla\cdot\vec{V})$

(vi) $\nabla \times (U\vec{V}) = U(\nabla \times \vec{V}) - \vec{V} \times (\nabla U)$

(vii) $\nabla\cdot(\vec{V} \times \vec{W}) = \vec{W}\cdot(\nabla \times \vec{V}) - \vec{V}\,(\nabla \times \vec{W})$

<u>Proof.</u> (i) – (v) are obvious. The sign "–" in (vi) is due to the order dependence in $\vec{V} \times (\nabla U) = -(\nabla U) \times \vec{V}$. Formula (vii) follows by developing the symbolic mixed product

$$\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \\ W_1 & W_2 & W_3 \end{vmatrix} = \begin{vmatrix} W_1 & W_2 & W_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} - \begin{vmatrix} V_1 & V_2 & V_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ W_1 & W_2 & W_3 \end{vmatrix}$$

(even though such a formula is not valid for vectors). ◇

When we handle with $\nabla$ as a differential operator it is advisable to respect the following:

**1.10. <u>Rule.</u>** (Step 1.) $\nabla$ applied to a product gives two terms, in which it acts on a single factor. We usually mark this action by an arrow "↓", as for example in the above (iv):

$$\nabla(UT) = \nabla(\overset{\downarrow}{U}\,T) + \nabla(U\,\overset{\downarrow}{T}).$$

(Step 2.) Realize the action of $\nabla$, as indicated by arrows, e.g.

$$\nabla(\overset{\downarrow}{U}\,T) + \nabla(U\,\overset{\downarrow}{T}) = (\nabla\,\overset{\downarrow}{U})T + U(\nabla\,\overset{\downarrow}{T}).$$

(Step 3.) Let after $\nabla$ a single letter, which distinguishes the field on which it acts, such that the arrows are not necessary anymore. For example:

$$(\nabla\,\overset{\downarrow}{U})T = T(\nabla U).$$

Other important formulas involving $\nabla$ are formulated in the problems at the end of this paragraph. Here we mention only the $\nabla$ form of the main integral formulas (established in §4, chapter VIII).

**1.11. <u>Corollary.</u>** Under the conditions stated in theorem 2, § VIII.4, the *Gauss-Ostrogradski* formula takes the form

$$\iiint\limits_D (\nabla\cdot\vec{V})dxdydz = \iint\limits_S (\vec{V}\cdot\vec{n})dS$$

**1.12. <u>Corollary.</u>** If the hypothesis of theorem 6, § VIII.4, is satisfied, then the *Stokes* formula holds in the form

$$\iint\limits_S (\nabla \text{x} \vec{V})\cdot\vec{n}\,dS = \int\limits_\Gamma \vec{V}\cdot d\vec{r}\,.$$

These formulas are useful just for better understanding of the divergence and rotation of a vector field:

**1.13. <u>Remark.</u>** In the case of a scalar field we have two possibilities of defining the gradient of $U$ at $A \in \mathscr{D}$, namely

$$grad\ U = \frac{\partial U}{\partial x}(A)\vec{i} + \frac{\partial U}{\partial y}(A)\ \vec{j} + \frac{\partial U}{\partial z}(A)\ \vec{k}\,,$$

as in definition 13, § IV.2, and according to corollary 4, § IV.3,

$$grad\ U = \left(\frac{\partial U}{\partial \vec{n}}\right)\vec{n}\,,$$

where $\vec{n}$ is the unit normal at the level surface passing through $A$.

Obviously, the first definition is preferable in calculations, but it seems to depend on system coordinates. Only the second definition shows that the gradient of a scalar field is an intrinsic characteristic of the field.

Similarly, for vector fields, so far we have used only the coordinate dependent expressions of ***div*** and ***rot***, so there is a problem whether they depend or not on the system coordinates. The answer is that they don't and this property follows from:

**1.14. <u>Theorem.</u>** Let $\mathscr{D}$, $S$ and $\vec{V} : \mathscr{D} \to \mathbb{R}^3$ be as in *Gauss-Ostrogradski* 's theorem. Let us fix $A \in \mathscr{D}$, and consider a sequence of sub-domains $(\mathscr{D}_{\mathrm{m}})_{m\in\mathbb{N}}$ of $\mathscr{D}$, containing $A$, and satisfying, together with their frontiers $S_m$, the same conditions as $\mathscr{D}$ and $S$. If $v_m = \mu(\mathscr{D}_{\mathrm{m}})$ is the volume of $\mathscr{D}_{\mathrm{m}}$, and $d_m = diameter(\mathscr{D}_{\mathrm{m}}) = \sup\{\|x - y\| : x, y \in \mathscr{D}_{\mathrm{m}}\}$ tend to zero when $m \to \infty$, then

$$(div\,\vec{V})(A) = \lim_{m\to\infty} \frac{1}{v_m} \iint_{S_m} \vec{V} \cdot \vec{n}\; dS.$$

<u>Proof.</u> The *Gauss-Ostrogradski* formula is valid for each $\mathscr{D}_{\mathrm{m}}$, i.e.

$$\iiint_{D_m} (\nabla \cdot \vec{V})\, dxdydz = \iint_{S_m} (\vec{V} \cdot \vec{n})\, dS\,.$$

Applying the mean value theorem to the triple integral, we can find some points $A_m \in \mathscr{D}_{\mathrm{m}}$, such that

$$(div\,\vec{V})(A_m) \cdot v_m = \iint_{S_m} (\vec{V} \cdot \vec{n})\, dS\,.$$

There remains to use the continuity of $div\,\vec{V}$, which gives

$$(div\,\vec{V})(A) = \lim_{m\to\infty} (div\,\vec{V})(A_m),$$

and realize the same limit in the Gauss-Ostrogradski formula. ◇

**1.15. <u>Theorem.</u>** Let $\vec{V} \in C^1_{\mathbb{R}^3}(\mathscr{D})$ be a vector function and let us fix a point $A \in \mathscr{D}$ and a unit vector $\vec{n} \in T_A$. In the plane of normal $\vec{n}$, passing through $A$, we consider a sequence $(S_m)_{\mathrm{m}\in\mathbb{N}}$ of elementary surfaces of borders $\Gamma_{\mathrm{m}}$. If $a_m = \mu(S_m)$ are the areas of $S_m$, and $d_m = diameter(S_m) \to 0$ when $m \to \infty$, then the component of the rotation of $\vec{V}$ at $A$ in the direction of $\vec{n}$ is

$$(rot\,\vec{V})(A) \cdot \vec{n} = \lim_{m\to\infty} \frac{1}{a_m} \int_{\Gamma_m} \vec{V} \cdot d\vec{r}\,.$$

<u>Proof.</u> According to Stokes' formula for $S_m$ and $\Gamma_{\mathrm{m}}$, $m \in \mathbb{N}$, we have

$$\iint_{S} (rot\vec{V}) \cdot \vec{n}\, dS = \int_{\Gamma_m} \vec{V} \cdot d\vec{r}\,,$$

and using the mean theorem fo the above double integral, we obtain

$$a_m\,[(rot\,\vec{V})(A_m) \cdot \vec{n}\,] = \int_{\Gamma_m} \vec{V} \cdot d\vec{r}\,,$$

where $A_m \in S_m$. Since $\vec{V}$ is of class $C^1$, $rot\,\vec{V}$ is continuous, hence

$$(rot\,\vec{V})(A) \cdot \vec{n} = \lim_{m\to\infty} (\mathrm{rot}\,\vec{V})(A_m) \cdot \vec{n}\,.$$

Finally, we take the same limit in the Stokes formula. ◇

**1.16. <u>Corollary.</u>** ($div\ \vec{V}$)($A$) and ($rot\ \vec{V}$)($A$) are independent of the system coordinates (i.e. they are intrinsic elements of the field at $A$).

<u>Proof.</u> The elements which appear in the right side of the relations established in the above theorems 14 and 15, as well as the volumes, areas, line integrals and surface integrals, are all independent of the system coordinates. ◇

In particular, we obtain the components of $rot\ \vec{V}$ on the canonical basis, if we consider that are obtained; $\vec{n}$ is successively equal to $\vec{i}$, $\vec{j}$ and $\vec{k}$, i.e.

$$(rot\ \vec{V})(A) \cdot \vec{i} = \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right)(A), \text{ etc.}$$

**PROBLEMS § IX.1**

**1.** Using rule 1.10, prove the formulas:

(i) $rot\,(U\vec{V}) = U\,rot\,\vec{V} - \vec{V} \times grad\,U$

(ii) $div(\vec{V} \times \vec{W}) = \vec{W}\,rot\,\vec{V} - \vec{V}\,rot\,\vec{W}$

(iii) $rot\,(\vec{V} \times \vec{W}) = \dfrac{\partial\vec{V}}{\partial\vec{W}} - \dfrac{\partial\vec{W}}{\partial\vec{V}} + \vec{V}\,div\,\vec{W} - \vec{W}\,div\,\vec{V}$

(iv) $grad\,(\vec{V}\,\vec{W}) = \vec{V} \times rot\,\vec{W} + \vec{W} \times rot\,\vec{V} + \dfrac{\partial\vec{W}}{\partial\vec{V}} + \dfrac{\partial\vec{V}}{\partial\vec{W}}$

(v) $\dfrac{\partial(U\vec{V})}{\partial\vec{l}} = \vec{V}\,(\vec{l}\,grad\,U) + U\,\dfrac{\partial\vec{V}}{\partial\vec{l}}$

<u>Hint.</u> (i) $\nabla \times (U\,\vec{V}) = \nabla \times (\overset{\downarrow}{U}\,\vec{V}) + \nabla \times (U\,\overset{\downarrow}{\vec{V}}) = (\nabla\,\overset{\downarrow}{U}) \times \vec{V} + U\,(\nabla \times \vec{V}) =$
$= U\,(\nabla \times \vec{V}) - \vec{V} \times (\nabla U)$.

(ii) $\nabla(\vec{V} \times \vec{W}) = \nabla(\overset{\downarrow}{\vec{V}} \times \vec{W}) + \nabla(\vec{V} \times \overset{\downarrow}{\vec{W}}) = \vec{W}\,(\nabla \times \vec{V}) - \vec{V}\,(\nabla \times \vec{W})$, where the last equality expresses the rule of interchanging the factors in a mixed product (compare with (vi) and (vii) in proposition 1.9).

(iv) $\nabla \times (\vec{V} \times \vec{W}) = \nabla \times (\overset{\downarrow}{\vec{V}} \times \vec{W}) + \nabla \times (\vec{V} \times \overset{\downarrow}{\vec{W}})$, where use

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}\,\vec{c})\vec{b} - (\vec{a}\,\vec{b})\vec{c},$$

we obtain

$$\nabla \times (\overset{\downarrow}{\vec{V}} \times \vec{W}) = (\nabla\,\vec{W})\overset{\downarrow}{\vec{V}} - (\nabla\,\overset{\downarrow}{\vec{V}})\vec{W} = \frac{\partial\vec{V}}{\partial\vec{W}} - \vec{W}\,div\,\vec{V}, \text{ etc.}$$

(iv) The relation $\nabla(\vec{V}\,\vec{W}) = \nabla(\overset{\downarrow}{\vec{V}}\,\vec{W}) + \nabla(\vec{V}\,\overset{\downarrow}{\vec{W}})$ is not to be continued by $\vec{W}\,div\,\vec{V} + \vec{V}\,div\,\vec{W}$, since the scalar product would be neglected. Using again the identity $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}\,\vec{c})\vec{b} - (\vec{a}\,\vec{b})\vec{c}$, in the sense that

$$\vec{V} \times (\nabla \times \overset{\downarrow}{\vec{W}}) = =\nabla(\vec{V}\,\overset{\downarrow}{\vec{W}}) - (\vec{V}\,\nabla)\overset{\downarrow}{\vec{W}},$$

we obtain

$$\nabla(\vec{V}\,\overset{\downarrow}{\vec{W}}) = \vec{V} \times (rot\;\vec{W}) + \frac{\partial\vec{W}}{\partial\vec{V}}.$$

($v$) $(\vec{l}\,\nabla)(U\,\vec{V}) = (\vec{l}\,\nabla)(\overset{\downarrow}{U}\,\vec{V}) + (\vec{l}\,\nabla)(U\,\overset{\downarrow}{\vec{V}}) = \vec{V}\,\dfrac{\partial U}{\partial\vec{l}} + U\,\dfrac{\partial\vec{V}}{\partial\vec{l}}$.

**2.** Let $\vec{a} \in \mathbb{R}^3$ be a fixed unit vector ($\|\vec{a}\| = 1$), and $\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$. Show that:

(i) $\frac{\partial \vec{r}}{\partial \vec{a}} = \vec{a}$

(ii) $\vec{a}$ [ $grad\,(\vec{V}\ \vec{a}) - rot\,(\vec{V} \times \vec{a})] = div\,\vec{V}$ (write it also for $\vec{V} = \vec{r}$ )

(iii) $div\,[\|\vec{r}\|\,(\vec{a} \times \vec{r})] = 0$.

Hint. (ii) Express $rot\,(\vec{V} \times \vec{a})$ according to problem 1 (iii), and multiply by $\vec{a}$. (iii) Combine proposition 1.9, (*v*) and problem 1 (ii).

**3.** Show that if $\mathscr{D} \subset \mathbb{R}^3$ is a regular domain, and $u, v \in C^2_{\mathbb{R}}(\mathscr{D})$, then the following Green's formula holds:

$$\iiint\limits_D (u\Delta v - v\Delta u)\,d\Omega = \iint\limits_S \left( u\frac{\partial v}{\partial \vec{n}} - v\frac{\partial u}{\partial \vec{n}} \right) dS,$$

where $\vec{n}$ is the unit normal to $S$ at its current point.

Hint. Write the *Gauss-Ostrogradski* formula for $\vec{V} = u\ grad\ v$ and $\vec{W} = v\ grad\ u$, and subtract the forthcoming relations. We start with $div\,(u\ grad\ v) = grad\ u \cdot grad\ v + u \cdot \Delta v$, then we introduce $\frac{\partial}{\partial \vec{n}}$.

**4.** Let $\mathscr{D} \subset \mathbb{R}^3$ be a regular domain of frontier $S$, and $\vec{V} \in C^2_{\mathbb{R}^3}(\mathscr{D})$. Show that

$$\iiint\limits_D rot\vec{V}\,d\Omega = \iint\limits_S \vec{n} \times \vec{V}\,dS\,,$$

where $\vec{n}$ is the unit normal at the current point of $S$, and the integrals of the vector functions are understood on components.

Hint. Apply the *Gauss-Ostrogradski* formula to $\vec{V} \times \vec{W}$, where $W$ is an arbitrary constant field. Since $\nabla \times \vec{W} = 0$, $\nabla(\vec{V} \times \vec{W}) = \vec{W} \cdot (\nabla \times \vec{V})$, it follows that

$$\iiint\limits_D div(\vec{V} \times \vec{W})\,d\Omega = \iiint\limits_D \vec{W} \cdot (rot\vec{V})\,dxdydz = \iint\limits_S (\vec{V} \times \vec{W}) \cdot \vec{n}\,dS = \iint\limits_S \vec{W} \cdot (\vec{n} \times \vec{V})\,dS\,.$$

Consequently for arbitrary $\vec{W}$ we have

$$\vec{W} \cdot \iiint\limits_D rot\vec{V}\,d\Omega = \vec{W} \cdot \iint\limits_S \vec{n} \times \vec{V}\,dS\,.$$

**5.** Evaluate the flux of the field $\vec{V} = \vec{a} \times \vec{r} + (\vec{a} \cdot \vec{r})\,\vec{a}$ through a closed surface $S$, where $\vec{r}$ is the position vector of the current point, and $\vec{a}$ is a constant unit vector.

Hint. Verify that $div\ \vec{V} = 1$, and apply the *Gauss - Ostrogradski* formula. The flux is equal to the volume bounded by $S$ .

**6.** Show that for any $\varphi,\ \psi \in C^2_{\mathbb{R}}(\mathscr{D})$, the field $\vec{V} = \varphi\, grad\ \psi + grad\ \varphi$ is orthogonal to $rot\ \vec{V}$.

Hint. Establish that $rot\ \vec{V} = grad\ \varphi \times grad\ \psi$ using problem 1, ($i$) .

**7.** Let $\vec{a} = (a_x , a_y , a_z )$ and $\vec{b} = (b_x , b_y , b_z )$ be constant vectors. We note $\varphi = \vec{a}\ grad\ r^3$, and $\psi = \vec{b}\ grad\ r^{-3}$, where $\vec{r}$ is the position vector of the current point. Show that:

(i) $\vec{b}\ grad\ \varphi + r^6\, \vec{a}\, grad\ \psi = \dfrac{18}{r}(\vec{a}\cdot\vec{r})(\vec{b}\cdot\vec{r})$

(ii) $\vec{b}\ grad\ div\,(\varphi\,\vec{r}) + \vec{a}\ grad\ div\,(r^6\,\psi\,\vec{r}) = 0$

(iii) $div\,(\varphi+\psi)\,\vec{r} - 3(\varphi+\psi) = 6r\,(\vec{a}\cdot\vec{r}) + 12r^{-5}(\vec{b}\cdot\vec{r})$.

Hint. (i) Establish the explicit expressions

$$\varphi = 3\,r\,(x\,a_x + y\,a_y + z\,a_z) = 3r\left(\vec{r}\cdot\vec{a}\right),$$

$$\psi = -3\,r^{-5}(x\,b_x + y\,b_y + z\,b_z) = -\frac{3}{r^5}\left(\vec{r}\cdot\vec{b}\right),$$

then evaluate

$$\vec{b}\ grad\ \varphi = \frac{3}{r}\left(\vec{r}\cdot\vec{a}\right)\left(\vec{r}\cdot\vec{b}\right)+3r\left(\vec{a}\cdot\vec{b}\right), \text{ and}$$

$$\vec{a}\ grad\ \psi = -\frac{3}{r^5}\left(\vec{a}\cdot\vec{b}\right)+\frac{15}{r^7}\left(\vec{r}\cdot\vec{a}\right)\left(\vec{r}\cdot\vec{b}\right).$$

(ii) From $div\,(\varphi\,\vec{r}) = 9r\left(\vec{a}\cdot\vec{r}\right)$ it follows that

$$\vec{b}\ grad\,[div\,(\varphi\,\vec{r})] = \frac{9}{r}\left(\vec{r}\cdot\vec{a}\right)\left(\vec{r}\cdot\vec{b}\right)+9r\left(\vec{a}\cdot\vec{b}\right).$$

Similarly, since $div\,(r^6\,\psi\,\vec{r}) = 9r\left(\vec{b}\cdot\vec{r}\right)$, we have

$$\vec{a}\ grad\ div\,(r^6\,\psi\,\vec{r}) = -9\left[\frac{1}{r}\left(\vec{r}\cdot\vec{a}\right)\left(\vec{r}\cdot\vec{b}\right)+r\left(\vec{a}\cdot\vec{b}\right)\right].$$

(iii) Start with $\varphi + \psi = 3r\left(\vec{r}\cdot\vec{a}\right)-\dfrac{3}{r^5}\left(\vec{r}\cdot\vec{b}\right)$, and deduce

$$div\,[(\varphi+\psi)\,\vec{r}\,] = 3\,(\varphi+\psi) + 6r\left(\vec{a}\cdot\vec{r}\right) + \frac{12}{r^5}\left(\vec{r}\cdot\vec{b}\right).$$

## § IX.2. CURVILINEAR COORDINATES

Even though the differential operators define intrinsic elements of the fields, in practice it is sometimes important to express these operators in other than Cartesian coordinates, eg. spherical or cylindrical.

**2.1. <u>Definition.</u>** Let $\mathscr{E} \subseteq \mathbb{R}^3$ be a domain, and $T : \mathscr{E} \to \mathbb{R}^3$ be a vector function. We say that $T$ is a *coordinates change* (*transformation*) iff it is a 1:1 diffeomorphism between $\mathscr{E}$ and $\mathscr{D} = T(\mathscr{E})$, such that $\operatorname{Det} J_T > 0$ at any $(u, v, w) \in \mathscr{E}$. The surface

$$S_{u_0} = \{(u, v, w) \in \mathbb{R}^3 : u = u_0\}$$

is called *coordinate surface* of type $u$–constant; similarly are defined the coordinate surfaces $v$– constant, and $w$–constant. The curve

$$\gamma_u = S_{v_0} \cap S_{w_0}$$

is called *coordinate curve* of parameter $u$; similarly are defined the coordinate curves of parameters $v$ and $w$ .

The *unit normal vectors* to the surfaces $S_{u_0}$, $S_{v_0}$ and $S_{w_0}$ will be denoted $\vec{n}_1$, $\vec{n}_2$ respectively $\vec{n}_3$. The *unit tangent vectors* to the curves $\gamma_u$, $\gamma_v$, $\gamma_w$ are denoted by $\vec{l}_1$, $\vec{l}_2$, $\vec{l}_3$.

**2.2. <u>Remarks.</u>** The coordinates $u$, $v$, $w$ are usually called *curvilinear* because the coordinate curves are not straight lines as in the case of the Cartesian coordinates. The change of coordinates can also be expressed by the correspondence between curvilinear and Cartesian coordinates

$$\mathscr{E} \ni (u, v, w) \xrightarrow{T} (x, y, z) \in \mathscr{D},$$

which is explicitly written using the components $f$, $g$, $h$ of $T$, i.e.

$$\begin{cases} x = f(u, v, w) \\ y = g(u, v, w) \\ z = h(u, v, w). \end{cases}$$

These formulas furnish the Cartesian equations of the coordinates surfaces and coordinates curves.

**2.3. <u>Examples.</u>** (i) The spherical coordinates $(\rho, \varphi, \theta)$ are introduced by

$$\begin{cases} x = \rho \sin\theta \cos\varphi \\ y = \rho \sin\theta \sin\varphi \\ z = \rho \cos\theta \end{cases}$$

hence $S_{\rho_0}$ is a sphere, $S_{\theta_0}$ is a cone and $S_{\varphi_0}$ is a half-plane.

Consequently $\gamma_\rho$ is a half-line, $\gamma_\theta$ is a half-circle, and $\gamma_\varphi$ is a circle (see Fig. IX.2.1, a)).

The vectors $\vec{n}_1$, $\vec{n}_2$ and $\vec{n}_3$ are orthogonal to each other, and $\vec{n}_k = \vec{l}_k$ for all $k$ = 1, 2, 3.

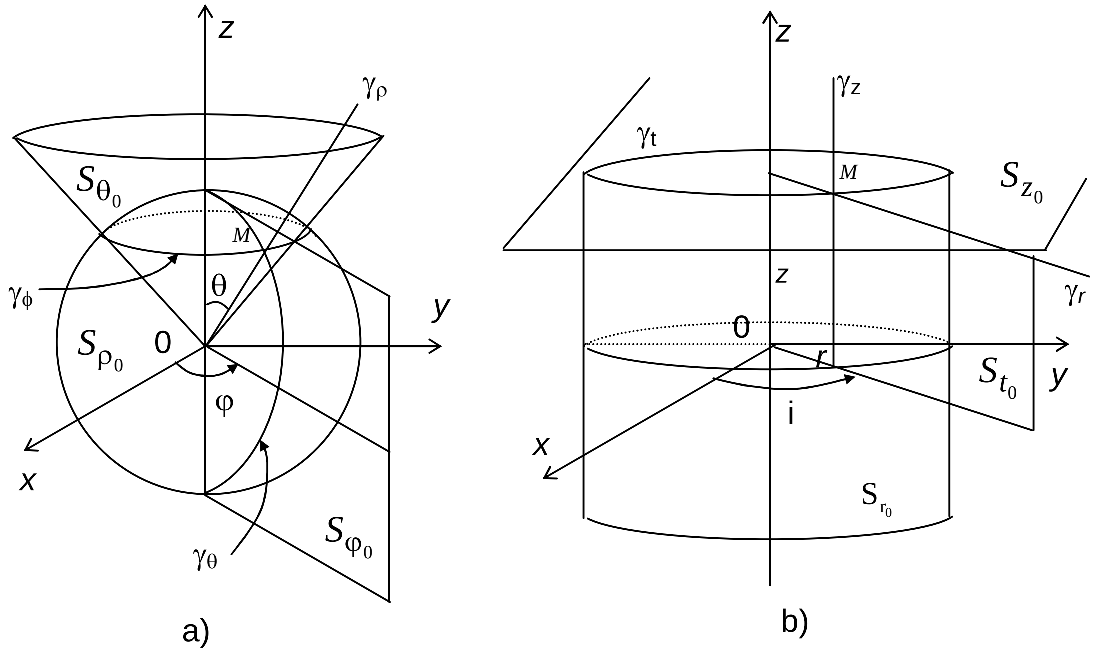



Fig. IX.2.1

(ii) The cylindrical coordinates ($r$, $t$, $z$) are defined by

$$\begin{cases} x = r\cos t \\ y = r\sin t \\ z = z, \end{cases}$$

hence $S_{r_0}$ is a cylinder, $S_{t_0}$ is a half-plane and $S_{z_0}$ is a plane, respectively $\gamma_r$ is a half-line, $\gamma_t$ is a circle and $\gamma_z$ is a straight line (Fig. IX.2.1, b)).

Similarly, $\{\vec{n}_1, \vec{n}_2, \vec{n}_3\}$ is a system of orthogonal vectors, and $\vec{n}_k = \vec{l}_k$ for all $k$ = 1, 2, 3.

(iii) Generally speaking $\{\vec{n}_1, \vec{n}_2, \vec{n}_3\}$ and $\{\vec{l}_1, \vec{l}_2, \vec{l}_3\}$ are not orthogonal systems of vectors, and $\vec{n}_k \neq \vec{l}_k$, for some $k$ = 1, 2, 3. For example we can consider the coordinates ($u$, $v$, $w$) defined by

$$\begin{cases} x = sh\,u + ch\,v \\ y = ch\,u + sh\,v \\ z = w. \end{cases}$$

However, there are strong relations between these vectors:

**2.4. <u>Proposition.</u>** (i) Noting $\vec{r}_u = \frac{\partial f}{\partial u}\vec{i} + \frac{\partial g}{\partial u}\vec{j} + \frac{\partial h}{\partial u}\vec{k}$, etc; we have $\vec{r}_u \parallel \vec{l}_1$, $\vec{r}_v \parallel \vec{l}_2$ and $\vec{r}_w \parallel \vec{l}_3$.

(ii) If, reversing $T$, we note

$$\begin{cases} u = \varphi(x, y, z) \\ v = \psi(x, y, z) \\ w = \chi(x, y, z) \end{cases}$$

then $\vec{n}_1 \parallel grad\ \varphi$, $\vec{n}_2 \parallel grad\ \psi$, and $\vec{n}_3 \parallel grad\ \chi$.

(iii) $\{grad\ \varphi,\ grad\ \psi,\ grad\ \chi\}$ and $\{\vec{r}_u, \vec{r}_v, \vec{r}_w\}$ are reciprocal systems of vectors, i.e. $grad\ \varphi \cdot \vec{r}_u = 1$, $grad\ \varphi \cdot \vec{r}_v = 0$, $grad\ \varphi \cdot \vec{r}_w = 0$, etc.

Proof. (i) and (ii) are direct consequences of the definitions of a gradient and of a tangent to a curve. The relations (iii) express the fact that $T^{-1} \circ T = \iota$, hence their Jacobian matrices verify $\boldsymbol{J_T}^{-1} \cdot \boldsymbol{J_T} = \boldsymbol{I}$, where $I$ is the unit matrix. Since $T$ is non-degenerate, $\{\vec{n}_1, \vec{n}_2, \vec{n}_3\}$ and $\{\vec{l}_1, \vec{l}_2, \vec{l}_3\}$ are linearly independent. ◇

**2.5. Definition.** If $T$ is a change of parameters, then the numbers

$$\|\vec{r}_u\| = L_1,\ \|\vec{r}_v\| = L_2 \text{ and } \|\vec{r}_w\| = L_3$$

are called *Lamé parameters* and

$$\|grad\ \varphi\| = H_1,\ \|grad\ \psi\| = H_2 \text{ and } \|grad\ \chi\| = H_3$$

are called *differential coefficients of the first order*.

Using the previous proposition we deduce:

**2.6. Proposition.** (i) $\vec{r}_u = L_1\vec{l}_1$, $\vec{r}_v = L_2\vec{l}_2$, $\vec{r}_w = L_3\vec{l}_3$;

(ii) $grad\ \varphi = H_1\vec{n}_1$, $grad\ \psi = H_2\vec{n}_2$, $grad\ \chi = H_3\vec{n}_3$;

(iii) $L_kH_k = 1$ for all $k = 1, 2, 3$, whenever the system of coordinates $u, v, w$ is orthogonal (i.e. $\{\vec{l}_1, \vec{l}_2, \vec{l}_3\}$ is an orthogonal basis in $\mathbb{R}^3$).

Proof. (i) and (ii) is based on the fact that $\|\vec{l}_k\| = \|\vec{n}_k\| = 1$ for all $k = 1, 2, 3$. Relations (iii) are consequences of $grad\ \varphi \cdot \vec{r}_u = 1$, etc. ◇

**2.7. Examples.** (i) In spherical coordinates $(\rho, \varphi, \theta)$ we have

$L_1 = H_1 = = \|\vec{r}_\rho\| = 1$, $L_2 = \dfrac{1}{H_2} = \|\vec{r}_\varphi\| = \rho \sin\theta$, and $L_3 = \dfrac{1}{H_3} = \|\vec{r}_\theta\| = \rho$.

(ii) In cylindrical coordinates $(r, t, z)$ we have

$$L_1 = H_1 = \|\vec{r}_r\| = 1,\ L_2 = \frac{1}{H_2} = \|\vec{r}_t\| = r, \text{ and } L_3 = H_3 = \|\vec{r}_z\| = 1.$$

(iii) For $(u, v, w)$ in the above example 3, (iii), we have

$$L_1 = \|\vec{r}_u\| = = \sqrt{ch\,2u},\ L_2 = \|\vec{r}_v\| = \sqrt{ch\,2v}, \text{ and } \|\vec{r}_w\| = 1 = L_3 = H_3,$$

but

$$H_1 = \|grad\ \varphi\| = = \frac{\sqrt{ch\,2v}}{ch(u - v)} \text{ and } H_2 = \|grad\ \psi\| = \frac{\sqrt{ch\,2u}}{ch(u - v)}.$$

Consequently,

$$L_1H_1 \neq 1 \neq L_2H_2 \text{ even though } grad\ \varphi \cdot \vec{r}_u = 1 \text{ and } grad\ \psi \cdot \vec{r}_v = 1.$$

We mention that *grad* φ and *grad* ψ are easily obtained without making φ and ψ explicit, but calculating $\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial y}$ from systems of the form

$$\begin{cases} 1 = \operatorname{ch} u \dfrac{\partial\varphi}{\partial x} + \operatorname{sh} v \dfrac{\partial\psi}{\partial x} \\ 0 = \operatorname{sh} u \dfrac{\partial\varphi}{\partial x} + \operatorname{ch} v \dfrac{\partial\psi}{\partial x}. \end{cases}$$

We obtained them by deriving the initial relations relative to $x$ and $y$.

From now on, we will consider only orthogonal coordinates.

In order for us to use the Lamé parameters in writing the differential operators of a field, we need the expressions of the length, area and volume in orthogonal curvilinear coordinates.

**2.8. <u>Lemma.</u>** Let $(u, v, w)$ be an orthogonal system of coordinates, and let $L_1, L_2, L_3$ be the corresponding Lamé parameters.

(i) If $\gamma$ is a curve of parameterization $u = u(t)$, $v = v(t)$, $w = w(t)$, where $t \in [a, b]$, then the length of $\gamma$ is

$$\int_\gamma \| d\vec{r} \| = \int_a^b \sqrt{(L_1 u')^2 + (L_2 v')^2 + (L_3 w')^2}\, dt;$$

(ii) If $D$ is a measurable domain in the surface $S_{w_0}$, then the area of $D$ is

$$\iint_D dS = \iint_E L_1 L_2 du dv,$$

where $T(E) = D$;

(iii) If $\mathscr{D}$ is a measurable domain in $\mathbb{R}^3$, and $\mathscr{D} = T(\mathscr{E})$, then the volume of $\mathscr{D}$ is

$$\iiint_{\mathscr{D}} d\Omega = \iiint_{\mathscr{E}} L_1 L_2 L_3 du dv dw.$$

<u>Proof.</u> (i) We have

$$d\vec{r} = \vec{r}_u du + \vec{r}_v dv + \vec{r}_w dw = [(L_1 u')\vec{l}_1 + (L_2 v')\vec{l}_2 + +(L_3 w')\vec{l}_3]dt,$$

and because $\{\vec{l}_1, \vec{l}_2, \vec{l}_3\}$ is an orthogonal basis, it follows the corresponding formula of $\|d\vec{r}\|$.

(ii) Changing the variables $(x, y, z) \mapsto (u, v, w)$ in the surface integral on $D$, we replace $dS = \|\vec{r}_u \times \vec{r}_v\| du\, dv$, but since $\vec{r}_u \perp \vec{r}_v$, we have

$$\|\vec{r}_u \times \vec{r}_v\| = L_1 L_2.$$

(iii) Changing the variables $(x, y, z) \mapsto (u, v, w)$ in the triple integral on $\mathscr{D}$, we replace $d\Omega = \operatorname{Det} J_T\, du\, dv\, dw$, where $\operatorname{Det} J_T = \vec{r}_u\, (\vec{r}_v \times \vec{r}_w)$. ◇

The expressions of the differential operators in (orthogonal) curvilinear coordinates will be obtained starting out with their invariant definitions:

**2.9. Theorem.** Let $U : \mathscr{D} \to \mathbb{R}$ be a scalar field, $T : \mathscr{E} \to \mathscr{D}$ be a change of coordinates and let $\tilde{U} = U \circ T$ be the same field in the coordinates $(u, v, w)$ of $\mathscr{E}$. If $L_1$, $L_2$, $L_3$ are the Lamé parameters, then

$$grad\ \tilde{U} = \frac{1}{L_1}\frac{\partial \tilde{U}}{\partial u}\vec{l}_1 + \frac{1}{L_2}\frac{\partial \tilde{U}}{\partial v}\vec{l}_2 + \frac{1}{L_3}\frac{\partial \tilde{U}}{\partial w}\vec{l}_3.$$

Proof. According to the invariant definition of a gradient, the derivative into any direction $\vec{l}$ is the projection of the gradient on this direction, hence in particular

$$\frac{\partial \tilde{U}}{\partial \vec{l}_k} = (grad\ \tilde{U})\vec{l}_k \text{ for all } k = 1, 2, 3.$$

On the other hand , if $M_0 = (u_0, v_0, w_0)$ is fixed in $\mathscr{E}$, also by definition

$$\frac{\partial \tilde{U}}{\partial \vec{l}_1}(M_0) = \lim_{\Delta s \to 0} \frac{\tilde{U}(u_0 + \Delta u, v_0, w_0) - \tilde{U}(u_0, v_0, w_0)}{\Delta s},$$

where $\Delta s = \|\vec{r}_u\ (M_0)\|\Delta u$ is the distance between $M_0$ and $(u_0+\Delta u, v_0, w_0)$.

Consequently,

$$\frac{\partial \tilde{U}}{\partial \vec{l}_1} = \frac{1}{L_1}\frac{\partial \tilde{U}}{\partial u}$$

and similarly,

$$\frac{\partial \tilde{U}}{\partial \vec{l}_2} = \frac{1}{L_2}\frac{\partial \tilde{U}}{\partial v}, \qquad \frac{\partial \tilde{U}}{\partial \vec{l}_3} = \frac{1}{L_3}\frac{\partial \tilde{U}}{\partial w}.$$

From the components of $grad\ \tilde{U}$, relative to the basis $\{\vec{l}_1, \vec{l}_2, \vec{l}_3\}$, we immediately deduce the announced form of the vector $grad\ \tilde{U}$. ◇

**2.10. Theorem.** Let $\mathscr{D}$ be a regular domain in $\mathbb{R}^3$, $\vec{V} : \mathscr{D} \to \mathscr{T}$ a vector field, and let $\vec{W} = \vec{V} \circ T$, where $T : \mathscr{E} \to \mathscr{D}$ is a change of variables which has the Lamé parameters $L_1$, $L_2$, $L_3$. If

$W_1(u, v, w)$, $W_2(u, v, w)$, and $W_3(u, v, w)$

are the components of $\vec{W}$ in the local (orthogonal) basis $\{\vec{l}_1, \vec{l}_2, \vec{l}_3\}$, then

$$div\ \vec{W} = \frac{1}{L_1L_2L_3}\left[\frac{\partial(W_1L_2L_3)}{\partial u} + \frac{\partial(L_1W_2L_3)}{\partial v} + \frac{\partial(L_1L_2W_3)}{\partial w}\right].$$

Proof. Let us fix $M = (u_0, v_0, w_0)$ in $\mathscr{E}$, and let us consider a curvilinear paralleloid of boundary $S$ and volume $\Omega$ (as in Fig. IX.2.2), having the sides along the coordinate curves.

According to theorem 14, § IX.1, the invariant form of divergence is

$$div\ \vec{W}(M) = \lim_{\Omega \to 0} \frac{1}{\Omega}\iint_S \vec{W} \cdot \vec{n}\, dS.$$

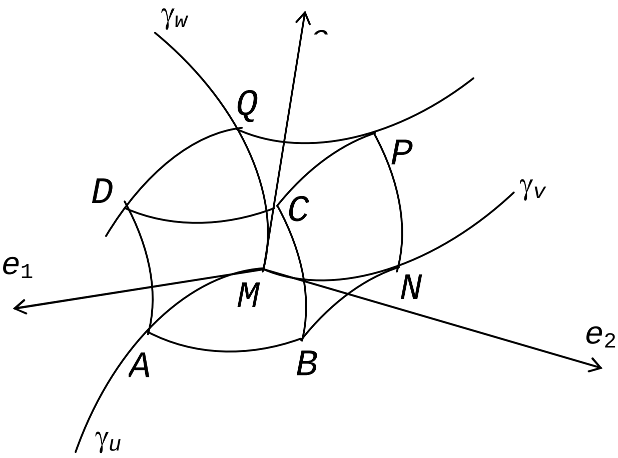



Fig. IX.2.2

To evaluate the flux of $\vec{W}$ through $S$ we calculate it for pairs of faces, e.g.

$$\Phi_1 = \iint_{ABCD} \vec{W}\cdot\vec{n}\,dS + \iint_{MNPQ} \vec{W}\cdot\vec{n}\,dS .$$

On the face $MNPQ$ we have $\vec{n} = -\vec{l}_1$, hence $\vec{W}\cdot\vec{n} = -W_1(u_0, v, w)$, and (approximately) the same on the face $ABCD$, i.e. $\vec{W}\cdot\vec{n} = W_1(u_0 + \Delta u, v, w)$. On both faces $dS = L_2L_3dvdw$, so that

$$\Phi_1 = \int_{v_0}^{v_0+\Delta v}\int_{w_0}^{w_0+\Delta w}[(W_1L_2L_3)(u_0+\Delta u, v, w) - (W_1L_2L_3)(u_0, v, w)]dvdw .$$

Using the Lagrange's theorem for the increment of $W_1L_2L_3$, and the mean-value theorem for the above double integral, we obtain

$$\Phi_1 = \int_{v_0}^{v_0+\Delta v}\int_{w_0}^{w_0+\Delta w}\frac{\partial(W_1L_2L_3)}{\partial u}(M_1')\Delta u dvdw = \frac{\partial(W_1L_2L_3)}{\partial u}(M_1^*)\Delta u\Delta v\Delta w ,$$

where $M'_1$ and $M_1^*$ are convenient points of the parallelepiped.

Similarly, there exist $M_2^*$ and $M_3^*$ in the parallelepiped, such that

$$\iint_S \vec{W}\cdot\vec{n}dS =$$

$$= \left[\frac{\partial(W_1L_2L_3)}{\partial u}(M_1^*) + \frac{\partial(L_1W_2L_3)}{\partial v}(M_2^*) + \frac{\partial(L_1L_2W_3)}{\partial w}(M_3^*)\right]\Delta u\Delta v\Delta w$$

On the other hand, $\Omega = L_1L_2L_3\ \Delta u\Delta v\Delta w$, hence it remains to use the continuity of $L_1$, $L_2$, $L_3$ and $W_1$, $W_2$, $W_3$. ◇

**2.11. <u>Corollary.</u>** Under the conditions of the above theorems (2.9 and 2.10), the Laplace operator of a scalar field $\tilde{U}$, in orthogonal curvilinear coordinates $(u, v, w)$ has the expression

$$\Delta\tilde{U} = \frac{1}{L_1L_2L_3}\left[\frac{\partial}{\partial u}\left(\frac{L_2L_3}{L_1}\cdot\frac{\partial\tilde{U}}{\partial u}\right)+\frac{\partial}{\partial v}\left(\frac{L_1L_3}{L_2}\cdot\frac{\partial\tilde{U}}{\partial v}\right)+\frac{\partial}{\partial w}\left(\frac{L_1L_2}{L_3}\cdot\frac{\partial\tilde{U}}{\partial w}\right)\right].$$

Proof. The components of $\vec{W} = grad\ \tilde{U}$ are $W_1 = \dfrac{1}{L_1}\dfrac{\partial\tilde{U}}{\partial u}$, $W_2 = \dfrac{1}{L_2}\dfrac{\partial\tilde{U}}{\partial v}$ and $W_3 = \dfrac{1}{L_3}\dfrac{\partial\tilde{U}}{\partial w}$, hence

$$W_1L_2L_3 = \frac{L_2L_3}{L_1}\frac{\partial\tilde{U}}{\partial u},\ L_1W_2L_3 = \frac{L_1L_3}{L_2}\frac{\partial\tilde{U}}{\partial v} \text{ and } L_1L_2W_3 = \frac{L_1L_2}{L_3}\frac{\partial\tilde{U}}{\partial w}.$$

Finally, $\Delta\tilde{U} = div\ \vec{W}$. ◇

**2.12. Theorem.** Using the above notations, under the conditions of Stokes' theorem, in orthogonal coordinates $(u, v, w)$, we have:

$$rot\ \vec{W} = \frac{1}{L_1L_2L_3}\begin{vmatrix} L_1\vec{l}_1 & L_2\vec{l}_2 & L_3\vec{l}_3 \\ \dfrac{\partial}{\partial u} & \dfrac{\partial}{\partial v} & \dfrac{\partial}{\partial w} \\ L_1W_1 & L_2W_2 & L_3W_3 \end{vmatrix}.$$

Proof. Aiming to find the component into direction $\vec{l}_1$ at $M = (u_0, v_0, w_0) \in \mathscr{E}$, we consider the surface $S$ in $S_{u_0}$ coordinate surface, bounded by the curvilinear rectangle $\overline{MNPQ} = \Gamma$ (see Fig. IX.2.3).

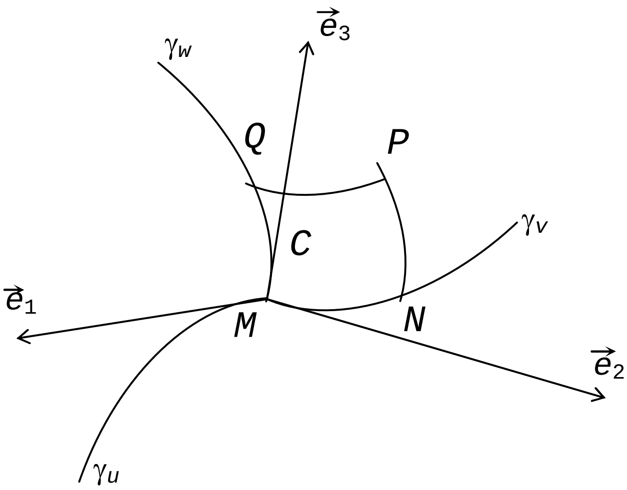



Fig. IX.2.3.

According to theorem 15, § IX.1, this component of $rot\ \vec{W}$ is

$$rot\ \vec{W}\,|\,\vec{l}_1 = \lim_{a\to 0}\frac{1}{a}\oint_\Gamma \vec{W}\,d\vec{r},$$

where $a$ is the area of $S = (\Gamma)$. In order to evaluate the curl on $\Gamma$, we evaluate the line integral on each side of $\Gamma$, e.g.

$$\int_{\overline{MN}} \vec{W}\, d\vec{r} = \int_{v_0}^{v_0+\Delta v} (W_2 L_2)(u_0, v, w_0)\, dv$$

since on $\overline{MN}$, we have $d\vec{r} = L_2\, dv\, \vec{l}_2$, etc. Consequently,

$$\int_{\Gamma} \vec{W}\, d\vec{r} = \int_{v_0}^{v_0+\Delta v} [(W_2L_2)(u_0, v, w_0) - (W_2L_2)(u_0, v, w_0 + \Delta w)] dv +$$

$$+ \int_{w_0}^{w_0+\Delta w} [(W_3L_3)(u_0, v_0 + \Delta v, w) - (W_3L_3)(u_0, v_0, w)] dw.$$

Expressing the increments by the Lagrange formula, and using the mean theorem for the above integrals it follows that:

$$\int_{\Gamma} \vec{W}\, d\vec{r} = -\int_{v_0}^{v_0+\Delta v} \frac{\partial(W_2L_2)}{\partial w}(M'_1)\Delta w dv + \int_{w_0}^{w_0+\Delta w} \frac{\partial(W_3L_3)}{\partial v}(M'_2)\Delta v dw =$$

$$= \left[\frac{\partial(W_3L_3)}{\partial v}(M_2^*) - \frac{\partial(W_2L_2)}{\partial w}(M_1^*)\right]\Delta v \Delta w$$

Taking into account that $a = L_2L_3\Delta v\Delta w$, and that all the involved functions are continuous, we obtain

$$rot\ \vec{W} \,|\, \vec{l}_1(M) = \frac{1}{L_2L_3}\left[\frac{\partial(W_3L_3)}{\partial v} - \frac{\partial(W_2L_2)}{\partial w}\right](M).$$

Similarly, we find the other components of *rot* $\overline{W}$, hence we have

$$rot\ \vec{W} = \frac{1}{L_2L_3}\left[\frac{\partial(W_3L_3)}{\partial v} - \frac{\partial(W_2L_2)}{\partial w}\right]\vec{l}_1 +$$

$$+ \frac{1}{L_3L_1}\left[\frac{\partial(W_1L_1)}{\partial w} - \frac{\partial(W_3L_3)}{\partial u}\right]\vec{l}_2 + \frac{1}{L_1L_2}\left[\frac{\partial(W_2L_2)}{\partial u} - \frac{\partial(W_1L_1)}{\partial v}\right]\vec{l}_3$$

which formally can be written as the above determinant. ◇

The spherical and cylindrical coordinates are frequently most used.

**2.13. <u>Differential operators in spherical coordinates.</u>** If, in particular, we take $(u, v, w) = (\rho, \varphi, \theta)$, then $L_1 = 1$, $L_2 = \rho \sin\theta$, $L_3 = \rho$, and:

$$grad\ \tilde{U} = \frac{\partial \tilde{U}}{\partial \rho}\vec{l}_1 + \frac{1}{\rho \sin\theta}\frac{\partial \tilde{U}}{\partial \varphi}\vec{l}_2 + \frac{1}{\rho}\frac{\partial \tilde{U}}{\partial \theta}\vec{l}_3,$$

$$div\ \vec{W} = \frac{1}{\rho^2 \sin\theta}\left[\sin\theta \frac{\partial(\rho^2 W_1)}{\partial \rho} + \rho\frac{\partial W_2}{\partial \varphi} + \rho\frac{\partial(W_3 \sin\theta)}{\partial \theta}\right] =$$

$$= \frac{\partial W_1}{\partial \rho} + \frac{1}{\rho\sin\theta}\cdot\frac{\partial W_2}{\partial \varphi} + \frac{1}{\rho}\cdot\frac{\partial W_3}{\partial \theta} + \frac{1}{\rho}W_1 + \frac{\cos\theta}{\rho \sin\theta}W_3$$

$$rot\vec{W} = \begin{vmatrix} \vec{l}_1 & \rho\sin\theta\vec{l}_2 & \rho\vec{l}_3 \\ \dfrac{\partial}{\partial\rho} & \dfrac{\partial}{\partial\varphi} & \dfrac{\partial}{\partial\theta} \\ W_1 & W_2\rho\sin\theta & \rho W_3 \end{vmatrix} \frac{1}{\rho^2\sin\theta} =$$

$$= \frac{1}{\rho^2\sin\theta}\left\{\rho\left[\frac{\partial W_3}{\partial\varphi} - \frac{\partial(W_2\sin\theta)}{\partial\theta}\right]\vec{l}_1 + \right.$$

$$\left. + \rho\sin\theta\left[\frac{\partial W_1}{\partial\theta} - \frac{\partial(\rho W_3)}{\partial\rho}\right]\vec{l}_2 + \rho\left[\sin\theta\frac{\partial(\rho W_2)}{\partial\rho} - \frac{\partial W_1}{\partial\varphi}\right]\vec{l}_3\right\},$$

$$\Delta\tilde{U} = \frac{1}{\rho^2}\left[\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial\tilde{U}}{\partial\rho}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\tilde{U}}{\partial\varphi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\tilde{U}}{\partial\theta}\right)\right].$$

**2.14. <u>Differential operators in cylindrical coordinates.</u>** If $(u, v, w) = (r, t, z)$ are cylindrical coordinates, then $L_1=1$, $L_2 = r$, $L_3 = 1$, and

$$grad\ \tilde{U} = \frac{\partial\tilde{U}}{\partial r}\vec{l}_1 + \frac{1}{r}\frac{\partial\tilde{U}}{\partial t}\vec{l}_2 + \frac{\partial\tilde{U}}{\partial z}\vec{l}_3,$$

$$div\ \vec{W} = \frac{1}{r}\left[\frac{\partial(rW_1)}{\partial r} + \frac{\partial W_2}{\partial t} + r\frac{\partial W_3}{\partial z}\right],$$

$$rot\ \vec{W} = \frac{1}{r}\begin{vmatrix} \vec{l}_1 & r\vec{l}_2 & \vec{l}_3 \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial t} & \dfrac{\partial}{\partial z} \\ W_1 & rW_2 & W_3 \end{vmatrix} = \frac{1}{r}\left[\frac{\partial W_3}{\partial t} - r\frac{\partial W_2}{\partial z}\right]\vec{l}_1 +$$

$$+ \left[\frac{\partial W_1}{\partial z} - \frac{\partial W_3}{\partial r}\right]\vec{l}_2 + \frac{1}{r}\left[\frac{\partial(rW_2)}{\partial r} - \frac{\partial W_1}{\partial t}\right]\vec{l}_3,$$

$$\Delta\tilde{U} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\tilde{U}}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\tilde{U}}{\partial t^2} + \frac{\partial^2\tilde{U}}{\partial z^2}.$$

The proof reduces to a direct substitution, and it is left as an exercise.

**2.15. <u>Remark.</u>** The formalism based on a symbol like $\nabla$ is not possible anymore. In fact, the expression of *grad u* might suggest to consider

$$\tilde{\nabla} = \frac{1}{L_1}\frac{\partial}{\partial u}\vec{l}_1 + \frac{1}{L_2}\frac{\partial}{\partial v}\vec{l}_2 + \frac{1}{L_3}\frac{\partial}{\partial w}\vec{l}_3,$$

but obviously $div\ \vec{W} \neq \tilde{\nabla}\vec{W}$, and $rot\ \vec{W} \neq \tilde{\nabla} \times \vec{W}$, etc. Consequently, it is advisable to use $\nabla$ only to express differential operators in Cartesian coordinates.

## PROBLEMS § IX.2.

**1.** The *generalized* spherical and cylindrical coordinates are defined by

$$\begin{cases} x = a\rho\sin\theta\cos\varphi \\ y = b\rho\sin\theta\sin\varphi \\ z = c\rho\cos\theta \end{cases} \quad \text{and} \quad \begin{cases} x = ar\cos t \\ y = br\sin t \\ z = c \end{cases}$$

where $a, b, c \in \mathbb{R}^*_+$.

(i) Identify the coordinate surfaces and the coordinate curves

(ii) Find the vectors $\vec{n}_1$, $\vec{n}_2$ and $\vec{n}_3$; and $\vec{l}_1, \vec{l}_2, \vec{l}_3$, and check their orthogonality

(iii) Evaluate $L_1$, $L_2$, $L_3$, and $H_1$, $H_2$, $H_3$.

Hint. Compare with examples 2.3, 2.7 (*i*) and (*ii*) of this section.

**2.** Consider the (non-orthogonal) system of coordinates $(u, v, w)$ defined by $T : \mathbb{R}^3 \to \mathbb{R}^3$, where $(x, y, z) = T(u, v, w)$ means:

$$\begin{cases} x = sh\, u + ch\, v \\ y = ch\, u + sh\, v \\ \mathrm{z} = \mathrm{w} \end{cases}$$

(i) If $I$ is the straight line segment of end points $(u_0, v_0, z_0) = (1, 1, 1)$ and $(u_1, v_1, w_1) = (2, 2, 1)$, find the length of $\gamma = T(I)$ using the Lamé parameters

(ii) If $Q$ is the square of diagonal $I$, find the area of $S = T(Q)$

(iii) If $K$ is the cube of base $Q$, find the volume of $D = T(K)$ .

Hint. $\mathrm{L}_1 = \sqrt{ch\, 2u}$, $\mathrm{L}_2 = \sqrt{ch\, 2v}$, $L_3 = 1$. We have $x = y$ iff $u = v$, hence $\gamma$ is the straight line segment of endpoints $(e, e, 1)$ and $(e^2, e^2, 1)$; a parameterization of $\gamma$ is $x = e^t$ , $y = e^t$ , $z = 1$, $t \in [1, 2]$. Because the coordinate curves are not orthogonal, we have to apply the formulas

$$\int_I \sqrt{d\,\vec{r}\cdot d\,\vec{r}} = \text{the length of } \gamma \text{ ,}$$

$$\iint_Q \| \vec{r}_u \,\mathrm{x}\, \vec{r}_v \| \, dudv = \text{the area of } S\text{, and}$$

$$\iiint_K \vec{r}_u \,(\vec{r}_v \,\mathrm{x}\, \vec{r}_v) dudvdw = \text{the volume of } D$$

**3.** Let $U : \mathscr{D} \to \mathbb{R}$ be a scalar field, $T : \mathscr{E} \to \mathscr{D}$ be a change of coordinates, and let $T^{-1}$ be expressed by

$$\begin{cases} u = \varphi(x, y, z) \\ v = \psi(x, y, z) \\ w = \chi(x, y, z), \end{cases}$$

where $(x, y, z) \in \mathscr{D}$. Show that

$$grad\ \tilde{U} = \frac{\partial \tilde{U}}{\partial u} grad\ \varphi + \frac{\partial \tilde{U}}{\partial v} grad\ \psi + \frac{\partial \tilde{U}}{\partial w} grad\ \chi,$$

and use it in order to obtain the expression of $grad\ \tilde{U}$ in orthogonal curvilinear coordinates.

Hint. $grad\ \varphi = H_1 \vec{n}_1 = \frac{1}{L_1}\vec{l}_1$, etc.

**4.** Establish the formulas:

(i) $rot\ \vec{l}_k = \frac{1}{L_k} grad(L_k) \times \vec{l}_k$, $k = 1, 2, 3$

(ii) $rot\ \vec{W} = \sum_{k=1,2,3} grad(L_k W_k) \times \vec{l}_k$ .

Hint. (i) Since $grad\ \varphi = \frac{1}{L_1}\vec{l}_1$, it follows that $rot\ (\frac{1}{L_1}\vec{l}_1) = 0$. On the other hand $rot\,(\frac{1}{L_1}\vec{l}_1) = -\vec{l}_1 + grad\ \frac{1}{L_1} + \frac{1}{L_1} rot\ \vec{l}_1$, where $grad\ \frac{1}{L_1} = -\frac{1}{L_1^{\,2}} grad\, L_1$.

(ii) $rot\ \vec{W} = \sum_{k=1,2,3} rot(W_k \vec{l}_k)$, where

$$rot(W_1 \vec{l}_1) = -\vec{l}_1 \times grad\ W_1 + W_1 rot\ \vec{l}_1 =$$

$$= [grad\ W_1 + W_1 \frac{1}{L_1}(grad\ L_1)] \times \vec{l}_1 = \frac{1}{L_1} grad\ (L_1 W_1) \times \vec{l}_1, \text{ etc.}$$

**5.** Establish the formulas

$$div\ \vec{l}_1 = \frac{1}{L_2 L_3}(grad\ L_2 L_3)\vec{l}_1, \text{ etc.}$$

and use them in order to obtain $div\ \vec{W}$ in orthogonal coordinates.

Hint. $\vec{l}_1 = \vec{l}_2 \times \vec{l}_3$, hence $div\ \vec{l}_1 = -\vec{l}_2\ rot\ \vec{l}_3 + \vec{l}_3\ rot\ \vec{l}_2$, (see problem 1 (ii), in § IX.1), where we can use the previous problem , and the properties of a mixed product. Further,

$$div\ (W_1 \vec{l}_1) = \frac{1}{L_2 L_3}(grad\ W_1 L_2 L_3)\ \vec{l}_1 = \frac{1}{L_1 L_2 L_3} \cdot \frac{\partial (W_1 L_2 L_3)}{\partial u}, \text{ etc.}$$

**6.** Let us consider the system of coordinates ($u$, $v$, $w$), defined by

$$T:\ \begin{cases} x = \text{ch}\, u\ \cos\ v \\ y = \text{sh}\, u\ \sin\ v \\ z = w \end{cases}$$

Show that it is orthogonal, determine the Lamé coefficients, and write the Laplace equation in these coordinates.
Hint. $\vec{r}_u$, $\vec{r}_v$ and $\vec{r}_w$ form an orthogonal system of vectors at each point.

$L_1 = L_2 = \sqrt{ch^2 u - \cos^2 v}$, $L_3 = 1$. $\dfrac{\partial^2 \tilde{U}}{\partial u^2} + \dfrac{\partial^2 \tilde{U}}{\partial v^2} + (\text{ch}^2 u - \cos^2 v)\dfrac{\partial^2 \tilde{U}}{\partial w^2} = 0.$

**7.** Evaluate *div* $\vec{W}$ and *rot* $\vec{W}$ if, in spherical coordinates ($\rho$, $\varphi$, $\theta$), the field is defined by $\vec{W} = 3\rho^2\theta\vec{l}_1 + \rho^2\vec{l}_3$.
Solution. *div* $\vec{W} = 12\rho\theta + \rho\ \text{ctg}\ \theta$, *rot* $\vec{W} = 0$.

**8.** Find the potential from which derives $\vec{W} = 3\rho^2\theta\vec{l}_1 + \rho^2\vec{l}_3$ in spherical coordinates ($\vec{W}$ is non-rotational according to the previous problem 7).

Hint. $\tilde{U}(\rho, \varphi, \theta) = \displaystyle\int_{(\rho_0,\varphi_0,\theta_0)}^{(\rho,\varphi,\theta)} \vec{W}\cdot d\vec{r}$, where

$$d\vec{r} = \vec{r}_\rho d\rho + \vec{r}_\varphi d\varphi + \vec{r}_\theta d\theta = d\rho\vec{l}_1 + \rho\, sin\, \theta d\varphi\vec{l}_2 + \rho d\theta\vec{l}_3,$$

hence $\vec{W}d\vec{r} = 3\rho^2\theta d\rho + \rho^3 d\theta$. Using a particular line,

$$\tilde{U}(\rho, \varphi, \theta) = \int_{\rho_0}^{\rho} 3\rho^2\theta_0 d\rho + \int_{\theta_0}^{\theta} \rho^3 d\theta = \rho^3\theta + c.$$

**9.** Find the potentials of the following fields in cylindrical coordinates:
a) $\vec{V} = z\vec{l}_1 + r\,\vec{l}_3$ (*rot* $\vec{V} = 0$)
b) $\vec{W} = \dfrac{z}{r}\vec{l}_2 + t\,\vec{l}_3$ (*rot* $\vec{W} = 0$).

Hint. In cylindrical coordinates $d\vec{r} = \vec{l}_1 dr + r\vec{l}_2 dt + \vec{l}_3 dz$. In particular, $\vec{V}d\vec{r} = d(rz)$, hence $\tilde{U} = rz + const$. Similarly, $\vec{W}d\vec{r} = d(tz)$ implies $\tilde{\tilde{U}} = tz + const$.

**10.** In the local basis $\{l_1, l_2, l_3\}$ of the curvilinear coordinates ($u$, $v$, $w$) defined by

$$\begin{cases} x = \frac{1}{2}(u^2 - v^2) \\ y = uv \\ z = w \end{cases}$$

we consider the field $\vec{W} = u\vec{l}_1 + v\,\vec{l}_2 + w\,\vec{l}_3$. Show that *rot* $\vec{W} = 0$, and find the potential which generates $\vec{W}$.

<u>Hint.</u> $L_1 = L_2 = \sqrt{u^2 + v^2}$, $L_3 = 1$, hence

$$d\vec{r} = \sqrt{u^2+v^2}\,\vec{l}_1 du + \sqrt{u^2+v^2}\,\vec{l}_2 dv + \vec{l}_3 dw.$$

The potential

$$\tilde{U}(u, v, w) = \int_{(u_0,v_0,w_0)}^{(u,v,w)} u\sqrt{u^2+v^2}du + v\sqrt{u^2+v^2}dv + wdw$$

can be obtained using the formula

$$\int_{(u_0,v_0,w_0)}^{(u,v,w)} Pdu + Qdv + Rdw =$$

$$= \int_{u_0}^{u} P(t, v_0, w_0)dt + \int_{v_0}^{v} Q(u, t, w_0)dt + \int_{w_0}^{w} R(u, v, t)dt$$

i.e. evaluating the circulation on a particular broken line up to a constant. The result is $\tilde{U}(u, v, w) = \frac{1}{3}(u^2 + v^2)^{3/2} + w$.

## § IX.3. PARTICULAR FIELDS

So far we have been studying only the *non-rotational* fields as a particular type of vector fields (see line integrals non-depending on the curve, finding a function when the partial derivatives are known, etc). The central result on non-rotational fields refers to the fact that they derive from potentials and these potentials, can be expressed as line integrals of the second type (circulations). The most representative example of a non-rotational field is the Newtonian one

$$\vec{V} = k\,\frac{1}{r^3}\vec{r}\,,$$

where $k$ depends on the units, and $\vec{r}$ is the position vector of the current point ($\vec{V} = grad\ U$, where $U = -\frac{k}{r}$). It is easy to see that $div\ \vec{V} = 0$ too, so we are led to analyze similarly other types of fields.

**3.1. <u>Definition.</u>** The field $\vec{V} : \mathscr{D} \to \mathscr{T}$ is said to be *solenoidal* iff $div\ \vec{V} = 0$ holds at each point of $\mathscr{D} \subseteq \mathbb{R}^3$.

Because of the practical meaning of $div\ \vec{V}$, expressed in terms of flux, the solenoidal fields are also called *fields without sources*.

**3.2. <u>Proposition.</u>** $\vec{V} : \mathscr{D} \to \mathscr{T}$ is a solenoidal field if and only if the flux of $\vec{V}$ through any closed surface $S$ is null (in the conditions of the *Gauss-Ostrogradski* theorem).

<u>Proof.</u> If $div\ \vec{V} = 0$, then $\iint_S \vec{V}\cdot\vec{n}\,dS = \iiint_\Omega div\ \vec{V} d\Omega = 0$ for any domain $\Omega$ with $Fr\ \Omega = S$. Consequently, we can use the invariant definition of $div\ \vec{V}$, specified in theorem 14, § IX.1). ◇

**3.3. <u>Theorem.</u>** The field $\vec{V} : \mathscr{D} \to \mathscr{T}$ is solenoidal if and only if for each $M_0 = (x_0, y_0, z_0) \in \mathscr{D}$ there exists a neighborhood $N \subseteq \mathscr{D}$, of $M_0$, and there exists a vector field $\vec{W} : N \to \mathscr{T}$ such that $\vec{V} = rot\ \vec{W}$ on $N$.

<u>Proof.</u> If $\vec{V} = rot\ \vec{W}$ then $div\ \vec{V} = \nabla\cdot(\nabla \times \vec{W}) = 0$. Conversely, let us choose $M_0 = (x_0, y_0, z_0) \in \mathscr{D}$ for which $N = S(M_0, r) \subseteq \mathscr{D}$ for some $r > 0$ (which exists because $\mathscr{D}$ is open). By hypothesis

$$div\ \vec{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z} = 0$$

on $\mathscr{D}$, which is valid on $N$ too. The problem is to construct the field $\vec{W}$, such that $\vec{V} = rot\ \vec{W}$ on $N$. We show that there exists such a field in the particular form $\vec{W} = W_1\vec{i} + W_2\vec{j}$, i.e. the following relation is possible:

$$\vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ W_1 & W_2 & 0 \end{vmatrix}.$$

In fact, the problem reduces to solving the system:

$$\begin{cases} \dfrac{\partial W_2}{\partial z} = -V_1 \\ \dfrac{\partial W_1}{\partial z} = V_2 \\ \dfrac{\partial W_2}{\partial x} - \dfrac{\partial W_1}{\partial y} = V_3 \end{cases} \qquad (*)$$

on $N$. The first equation gives

$$W_2(x, y, z) = -\int_{z_0}^{z} V_1(x, y, z)dt + \varphi(x, y),$$

where $\varphi$ is an arbitrary real function of class $C^1$ on $N$. Similarly, from the second equation we deduce

$$W_1(x, y, z) = \int_{z_0}^{z} V_2(x, y, z)dt + \psi(x, y).$$

Replacing $W_2$ and $W_3$ in the third equation we obtain

$$-\int_{z_0}^{z} \left( \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} \right)(x, y, t)dt + \frac{\partial \varphi}{\partial x}(x, y) - \frac{\partial \psi}{\partial y}(x, y) = V_3(x, y, z)$$

or, using the hypothesis that $div\,\vec{V} = 0$,

$$\int_{z_0}^{z} \frac{\partial V_3}{\partial z}(x, y, t)dt + \frac{\partial \varphi}{\partial x}(x, y) - \frac{\partial \psi}{\partial y}(x, y) = V_3(x, y, z).$$

Applying the Leibniz-Newton formula to the above integral, it follows

$$\frac{\partial \varphi}{\partial x}(x, y) - \frac{\partial \psi}{\partial y}(x, y) = V_3(x, y, z_0).$$

Obviously, there are functions $\varphi$ and $\psi$ satisfying this condition, hence $W_1$ and $W_2$ are completely (but not uniquely) determined. ◇

**3.4. <u>Remark.</u>** The construction of $\vec{W}$ by solving (*) also represents the practical method of solving problems in which $\vec{W}$ is asked. Usually, the method furnishes $\vec{W}$ on the whole $\mathscr{D}$, even though the proof is restricted to some neighborhoods of the points at $\mathscr{D}$. If the construction of $\vec{W}$ must be realized repeatedly at different points, there arises the problem of comparing the fields on the common parts of the corresponding neighborhoods. This problem is solved by:

**3.5. <u>Proposition.</u>** A necessary and sufficient condition for the fields $\vec{W}$ and $\vec{Z}$ to verify the relations *rot* $\vec{W} = \vec{V} =$ *rot* $\vec{Z}$ on the open and star-like domain $\mathscr{D}$ is that

$$\vec{W} - \vec{Z} = grad\, U$$

for some scalar field $U$ on $\mathscr{D}$, and *rot* $\vec{W} = \vec{V}$.

<u>Proof.</u> If *rot* $\vec{W} = \vec{V} =$ *rot* $\vec{Z}$, then *rot* $(\vec{W} - \vec{Z}) = 0$, hence $\vec{W} - \vec{Z}$ derive from a potential $U$.

Conversely, if $\vec{V} =$ *rot* $\vec{W}$ and $\vec{Z} = \vec{W} + grad\, U$, then *rot* $\vec{Z} = \vec{V}$. ◇

By analogy to the case of the non-rotational fields, which are said to "derive" from a scalar potential, a similar terminology can be used for solenoidal fields in order to express theorem 3.3 from above.

**3.6. <u>Definition.</u>** The field $\vec{W}$, for which *rot* $\vec{W} = \vec{V}$, is called a *vector potential* of $\vec{V}$. If so, we also say that $\vec{V}$ *derives* form a vector potential $\vec{W}$.

Using these terms, the above results take the forms:

**3.7. <u>Corollary.</u>** (i) $\vec{V} : \mathscr{D} \to \mathscr{T}$ is a solenoidal field iff it locally derives from a vector potential.

(ii) Two vector potentials of the same field, on a star-like and open domain $\mathscr{D}$, differ by a gradient.

(iii) If $\vec{V}$ derives from the vector potential $\vec{W}$, and $S$ is a surface of border $\Gamma$ as in Stokes' theorem, then

$$\iint_S \vec{V} \cdot \vec{n}\, dS = \int_\Gamma \vec{W} \cdot d\,\vec{r}\,,$$

i.e. the flux of $\vec{V}$ through $S$ reduces to the circulation of $\vec{W}$ along $\Gamma$.

<u>Proof.</u> All these assertions represent reformulations of some previously established properties, namely theorem 3.3, proposition 3.5, and respectively the Stokes' theorem. ◇

As generalization of the potential fields we consider now another type of fields, which are generated by two scalar fields.

**3.8. <u>Definition.</u>** We say that the vector field $\vec{V} : \mathscr{D} \to \mathscr{T}$ is *bi-scalar* iff there exist two scalar fields $\varphi, \psi : \mathscr{D} \to \mathbb{R}$ such that

$$\vec{V} = \varphi\, grad\, \psi.$$

The bi-scalar fields can be characterized in terms of rotation.

**3.9. <u>Theorem.</u>** $\vec{V} : \mathscr{D} \to \mathscr{T}$ is a bi-scalar field iff $\vec{V}$ *rot* $\vec{V} = 0$.

<u>Proof.</u> If $\vec{V}$ is a bi-scalar field, then $\vec{V}$ *rot* $\vec{V} = 0$ since

$$rot\, \vec{V} = grad\, \varphi \times grad\, \psi.$$

Conversely, if $\vec{V}$ *rot* $\vec{V} = 0$, this is sufficient for the equation

$$V_1dx + V_2dy + V_3dz = 0 \qquad (**)$$

to have solution, where $V_1$, $V_2$ and $V_3$ are the components of $\vec{V}$.

In fact, this equation is equivalent to the system

$$\begin{cases} \dfrac{\partial z}{\partial x} = A(x, y, z) \\ \dfrac{\partial z}{\partial y} = B(x, y, z) \end{cases}$$

where $A = -\dfrac{V_1}{V_3}$ and $B = -\dfrac{V_2}{V_3}$, which is integrable iff

$$\frac{\partial A}{\partial y} + \frac{\partial A}{\partial z} B = \frac{\partial B}{\partial x} + \frac{\partial B}{\partial z} A,$$

i.e. $\vec{V}\, rot\ \vec{V} = 0$. This conditions assures the possibility of integrating successively the equations of the system, i.e. in $z = f(x, y) + C(x)$ obtained by integrating the second equation we can determine $C$ such that the first equation to be satisfied too. We say that $U$ be a solution of (**), if $U(x, y, z) = 0$ represents the implicit form of the solution $z = z(x, y)$. In this case there exists an integrand factor $\mu$ such that

$$\mu V_1dx + \mu V_2dy + \mu V_3\, dz = dU,$$

or, equivalently, $\mu\vec{V} = grad\ U$. In other notation, namely $\varphi = \dfrac{1}{\mu}$, and $\psi = U$, this means $\vec{V} = \varphi\ grad\ \psi$. ◇

Another characterization of the bi-scalar fields is formulated in the more geometrical terms involving the field lines:

**3.10. <u>Definition.</u>** The curve $\mathscr{L} \subseteq \mathscr{D}$ is called *field line* of $\vec{V} : \mathscr{D} \to \mathscr{T}$ iff $\vec{V}(M)$ is tangent to $\mathscr{L}$ at each $M \in \mathscr{L}$. According to this definition, the field lines are solutions of the system

$$\frac{dx}{V_1} = \frac{dy}{V_2} = \frac{dz}{V_3},$$

where two of the variables $x, y, z$ are searched as functions depending on the third one.

**3.11. <u>Theorem.</u>** $\vec{V} : \mathscr{D} \to \mathscr{T}$ is a bi-scalar field iff there exists a family of surfaces in $\mathscr{D}$, which are orthogonal to the field lines of $\vec{V}$.

<u>Proof.</u> If $\vec{V}$ is bi-scalar, then $\vec{V} \parallel grad\ \psi$. Therefore $\vec{V}$ is orthogonal to the surface $\psi = \psi(M_0)$, where $M_0 \in \mathscr{L}$, and $\vec{V}$ is tangent to $\mathscr{L}$ at $M_0$.

Conversely, if the field lines are orthogonal to the family of surfaces $\psi(x, y, z) = const.$, then $\vec{V} \parallel grad\ \psi$, hence $\vec{V} = \varphi\ grad\ \psi$. ◇

The following proposition introduces some of the most remarkable properties related to bi-scalar fields, also known as *Green formulas*.

**3.12. <u>Proposition.</u>** Let $\vec{V} = \varphi\, grad\, \psi$ be a bi-scalar field in $\mathscr{D} \subseteq \mathbb{R}^3$, and let $\Omega \subset \mathscr{D}$ be a regular domain of frontier $S$ (as in the Gauss-Ostrogradski theorem). If $\vec{n}$ is the unit normal to $S$ at the current point, then we have:

(i) $\displaystyle\iint_S \varphi \frac{\partial\psi}{\partial\vec{n}} dS = \iiint_\Omega [\varphi\Delta\psi + grad\,\varphi\, grad\,\psi] d\Omega$

(ii) $\displaystyle\iint_S \left[\varphi\frac{\partial\psi}{\partial\vec{n}} - \psi\frac{\partial\varphi}{\partial\vec{n}}\right] dS = \iiint_\Omega [\varphi\Delta\psi - \psi\Delta\varphi] d\Omega$

(iii) $\displaystyle\iint_S \frac{\partial\varphi}{\partial\vec{n}} dS = \iiint_\Omega \Delta\varphi d\Omega$.

<u>Proof.</u> (i) If we apply the Gauss-Ostrogradski theorem to $\vec{V} = \varphi\, grad\, \psi$, then we obtain $\vec{V}\cdot\vec{n} = \varphi\dfrac{\partial\psi}{\partial\vec{n}}$ and $div\,\vec{V} = \varphi\Delta\psi + grad\,\varphi\, grad\,\psi$.

(ii) We write (i) for $\vec{V} = \varphi\, grad\, \psi$ and $\vec{W} = \psi\, grad\, \varphi$, and subtract the corresponding formulas.

(iii) Take $\psi = 1$ in (ii), such that $\dfrac{\partial\psi}{\partial\vec{n}} = 0$, and $\Delta\psi = 0$. ◇

**3.13. <u>Remark.</u>** Condition $\vec{V}\, rot\, \vec{V} = 0$ is useful in practice in order to recognize the bi-scalar fields. The problem of writing a bi-scalar field in the form $\varphi\, grad\, \psi$ may be solved using theorem 3.11. In fact, solving the equation $\vec{V}\, d\vec{r} = 0$, we find the family of surfaces $\psi = const.$, which are orthogonal to $\vec{V}$, hence $\vec{V} \parallel grad\,\psi$. Finally, we identify $\varphi$ such that $\vec{V} = \varphi\, grad\, \psi$.

The last type of fields, which will be considered here, is that of the "harmonic" fields. Even though they are more particular than the previous ones, these fields are the object of a wide part of mathematics, called harmonic analysis.

**3.14. <u>Definition.</u>** Let $\mathscr{D}$ be a domain in $\mathbb{R}^3$, and $\vec{V} \in C^1_{R^3}(\mathscr{D})$. We say that $\vec{V}$ is a *harmonic* field iff it is simultaneously solenoidal and irrotational in $\mathscr{D}$. The scalar field $U : \mathscr{D} \to \mathbb{R}$ is said to be *harmonic* iff $\Delta U = 0$ (alternatively we can say that $U$ is a harmonic function).

A significant example of harmonic field is $\vec{V} = k\,\dfrac{\vec{r}}{r^3}$.

**3.15. <u>Theorem.</u>** Let $\mathscr{D}$ be an open and star-like set in $\mathbb{R}^3$, and let $\vec{V} : \mathscr{D} \to \mathscr{T}$ be a field of class $C^1$ on $\mathscr{D}$. Then $\vec{V}$ is harmonic on $\mathscr{D}$ iff there exists a scalar field $U : \mathscr{D} \to \mathbb{R}$, of class $C^2$ on $\mathscr{D}$, such that $\Delta U = 0$ and $\vec{V} = grad\, U$ in $\mathscr{D}$.

Proof. $\vec{V}$ is irrotational iff $\vec{V} = grad\ U$. On the other hand, $\vec{V} = grad\ U$ is solenoidal iff $\Delta U = 0$. ◇

The following theorem shows that the harmonic fields are determined by their values on the frontier of the considered domain.

**3.16. Theorem.** Let $\mathcal{D} \subseteq \mathbb{R}^3$ be open and star-like, and let $\Omega \subseteq \mathcal{D}$ be a regular compact domain, bounded by $S$.

(i) If the harmonic functions $U_1$ and $U_2$ are equal on $S$, then they are equal on $\Omega$.

(ii) If the harmonic vector fields $\vec{V}_1$ and $\vec{V}_2$ have equal components along the normal to $S$ (at each point, then $\vec{V}_1$ and $\vec{V}_2$ are equal on $\Omega$.

Proof. (i) For $U = U_2 - U_1$, we have $\Delta U = 0$ on $\Omega$, and $U|_S = 0$. If we note $\vec{V} = U\ grad\ U$, then

$$div\ \vec{V} = \|grad\ U\|^2 + U\Delta U = \|grad\ U\|^2,$$

and $\vec{V}\cdot\vec{n} = U\dfrac{\partial U}{\partial\vec{n}} = 0$ on $S$.

Consequently, according to the Gauss-Ostrogradski theorem, $\iiint\limits_{\Omega}\|gradU\|^2\,d\Omega = 0$, hence $grad\ U = 0$.

So we deduce that $U = constant$, and more exactly, $U = 0$ on $\mathcal{D}$ since $U|_S = 0$. In conclusion, $U_1 = U_2$ on $\Omega$.

(ii) Let us note $\vec{V} = \vec{V}_2 - \vec{V}_1$. Since $\vec{V}_1\cdot\vec{n} = \vec{V}_2\cdot\vec{n}$ on $S$, we deduce that $\vec{V}\cdot\vec{n} = 0$ on $S$. Because $\vec{V}$ is harmonic, there exists $U : \mathcal{D} \to \mathbb{R}$ such that $\vec{V} = grad\ U$. If we consider $\vec{W} = U\,\vec{V}$, it follows that

$$div\ \vec{W} = U\Delta U + \|grad\ U\|^2 = \|grad\ U\|^2$$

on $\mathcal{D}$, and $\vec{W}\cdot\vec{n} = U\vec{V}\cdot\vec{n} = 0$ on $S$. Using again the Gauss-Ostrogradski theorem, we obtain that

$$\iiint\limits_{\Omega}\|grad\ \mathrm{U}\|^2\,d\Omega = 0,$$

hence $grad\ U = 0$. Consequently, $\vec{V} = 0$ on $\mathcal{D}$, i.e. $\vec{V}_1 = \vec{V}_2$. ◇

**3.17. Remark.** The problem of finding the (unique) harmonic field, which is specified on the frontier, is specific to the theory of differential equations with partial derivatives of the second order. Without other details of this theory, we mention that a lot of properties of the harmonic fields are consequences of the previously established results concerning other particular fields. For example, from proposition 3.12, it follows that

$$\iint\limits_{S}\varphi\frac{\partial\psi}{\partial\vec{n}}\,dS = \iint\limits_{S}\psi\frac{\partial\varphi}{\partial\vec{n}}\,dS,$$

and

$$\iint_S \frac{\partial \varphi}{\partial \vec{n}} dS = 0$$

whenever φ and ψ are harmonic functions.

A problem which leads to the solution of the Poisson equation $\Delta\varphi = \lambda$ is that of determining a field of given rotation and divergence:

**3.18. <u>Proposition.</u>** The field $\vec{V} : \mathscr{D} \to \mathscr{T}$ for which $rot\ \vec{V} = \vec{a}$, $div\ \vec{V} = b$, where $\vec{a}$ is a given vector field, and $b$ are given scalar fields, is determined up to the gradient of a harmonic field.

<u>Proof.</u> We search for a solution of the form $\vec{V} = \vec{V}_1 + \vec{V}_2$, where

$$\begin{cases} rot\, \vec{V}_1 = 0 \\ div\, \vec{V}_1 = b \end{cases} \text{ and } \begin{cases} \mathrm{rot}\, \vec{V}_2 = \vec{a} \\ div\, \vec{V}_2 = 0 \end{cases}.$$

Because $\vec{V}_1$ is irrotational, there exists a scalar field φ on $\mathscr{D}$ such that $\vec{V}_1 = grad\ \varphi$. The second condition on $\vec{V}_1$ gives $\Delta\varphi = b$. If $\varphi_0$ denotes a particular solution of this equation, i.e. $\Delta\varphi_0 = b$, then $\varphi = \varphi_0 + \Phi$, where $\Delta\Phi = 0$. Consequently,

$$\vec{V}_1 = grad\ \varphi_0 + grad\ \Phi. \qquad (*)$$

Now, about $\vec{V}_1$, we remark that $div\ \vec{a} = div\ rot\ \vec{V}_2 = 0$, hence $\vec{a}$ is solenoidal, and $\vec{V}_2$ is a vector potential of $\vec{a}$. As usually, this potential is determined up to a gradient, i.e.

$$\vec{V}_2 = \vec{V}_0 + grad\ \psi,$$

where $\vec{V}_0$ is a particular vector potential of $\vec{a}$. On the other hand, because $div\ \vec{V}_2 = 0$, we obtain $\Delta\psi = -\ div\ \vec{V}_0$, which is another Poisson equation. If $\psi_0$ is a particular solution of this equation, then may write $\psi = \psi_0 + \Psi$, where $\Delta\Psi = 0$. Consequently,

$$\vec{V}_2 = \vec{V}_0 + grad\ \psi_0 + grad\ \Psi. \qquad (**)$$

Using (*) and (**) we obtain

$$\vec{V} = \vec{V}_0 + grad\ (\varphi_0 + \psi_0) + grad\ \Xi$$

where $\Xi = \Phi + \Psi$ is an arbitrary harmonic function. ◇

**3.19. <u>Remark.</u>** The way of proving the above proposition furnishes a practical method of finding a vector field for which we know the rotation and the divergence. The concrete determination of the left function Ξ is dependent on the form of $\mathscr{D}$, and of the values imposed on the frontier of $\mathscr{D}$. Solving the Laplace equation on $\mathscr{D}$ under given conditions on $Fr\ \mathscr{D}$ is a specific problem in the theory of differential equations with partial derivatives of the second order (see an appropriate bibliography).

## PROBLEMS § IX.3

**1.** Verify that the following fields are irrotational and find their potentials:

(i) $\vec{V} = (3y + x^2)\ \vec{i} + (2y^2 + 3xz)\vec{j} + (z^2 + 3xy)\vec{k}$ in Cartesian coordinates;

(ii) $\vec{F} = \dfrac{1}{\rho^2}\vec{l}_1$ in spherical coordinates;

(iii) $\vec{W} = 2r\,z\,sin\,t\ \vec{l}_1 + r\,z\,cos\,t\ \vec{l}_2 + r^2 sin\,t\ \vec{l}_3$ in cylindrical coordinates.

Hint. Evaluate the rotation in the corresponding coordinates and evaluate the line integrals of $\vec{V}\cdot d\vec{r}$ on particular broken curves (as in the previous paragraphs).

**2.** Show that the field $\vec{V} = r\,(\vec{\omega}\ \mathrm{x}\ \vec{r}\,)$ is solenoidal and find one of its vector potentials, where $\vec{\omega}$ is a constant (fixed) vector, $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ is the position vector of a current position in $\mathscr{D} = \mathbb{R}^3$, and $r = ||\vec{r}\,||$.

Hint. $div\ \vec{V} = (\nabla r)(\vec{\omega}\ \mathrm{x}\ \vec{r}\,) + r\nabla(\vec{\omega}\ \mathrm{x}\ \vec{r}\,) = 0 + r\vec{r}\ (\nabla\ \omega) - r\vec{\omega}(\nabla \mathrm{x}\ \vec{r}\,) = 0$, where $\omega = \|\vec{\omega}\|$, hence $\vec{V}$ is solenoidal. Since this property is intrinsic, we can choose the reference system such that $\vec{\omega}$ stays along the $z$–axis of a Cartesian system. Because $\vec{r}$ has an invariant form $x\vec{i} + y\vec{j} + z\vec{k}$, the problem reduces to find the vector potential of the field

$$\vec{V} = r\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 0 & 0 & \omega\\ x & y & z\end{vmatrix} = r\,\omega\,(-y\vec{i} + x\vec{j}\,).$$

Because looking for the vector potential of the form $\vec{W} = (W_1, W_2, 0)$ as in theorem 3.3, leads to inconvenient integrals, we may try other forms, e.g. $\vec{W} = (W_1, 0, W_3)$. In this case we have to integrate the system

$$\begin{cases}\dfrac{\partial W_3}{\partial y} = -r\,\omega\,y\\[2mm] \dfrac{\partial W_1}{\partial z} - \dfrac{\partial W_3}{\partial x} = r\,\omega\,x\\[2mm] \dfrac{\partial W_1}{\partial y} = 0.\end{cases}$$

We find $W_3 = -\frac{1}{z}\omega r^3 + \varphi(x, z)$, $W_1 = \psi(x, z)$ and according to the second equation, $\frac{\partial\psi}{\partial z} + \frac{\partial\varphi}{\partial x} = 0$. In particular, we can choose $\varphi = \psi = 0$, hence a vector potential is $\vec{W} = -\frac{1}{z}\omega r^3\,\vec{k} = -\frac{1}{z}r^3\vec{\omega}$.

**3.** Show that the following fields are solenoidal, but not irrotational, and determine a vector potential for each one:
(i) $\vec{V} = 2xy\vec{i} - y^2\vec{j} + \vec{k}$
(ii) $\vec{V} = x^2\vec{i} + xz\,\vec{j} - 2xz\vec{k}$
Hint. (i) Following theorem 3.3, we obtain

$$W_2 = -2xyz + \varphi(x, y), \quad W_1 = -y^2z + \psi(x, y),$$

where $\frac{\partial\varphi}{\partial x} + \frac{\partial\psi}{\partial y} = 1$; eg. $\varphi = 2x$, $\psi = y$.
(ii) Similarly, we find

$$W_2 = -x^2z + \varphi(x, y), \quad W_1 = \frac{1}{2}xz^2 + \psi(x, y),$$

where $\frac{\partial\varphi}{\partial x} - \frac{\partial\psi}{\partial y} = 0$, as for example $\varphi = \varphi(y)$ and $\psi = \psi(x)$.

**4.** Show that for every irrotational and solenoidal field $\vec{V}$, we have

$$grad\,(\vec{r}\,\vec{V}) + rot\,(\vec{r} \times \vec{V}) + \vec{V} = 0,$$

where $\vec{r}$ is the position vector of the current point.
Hint. $grad\,(\vec{r}\,\vec{V}) = \vec{r} \times rot\,\vec{V} + \vec{V} \times rot\,\vec{r} + \frac{\partial\vec{V}}{\partial\vec{r}} + \frac{\partial\vec{r}}{\partial\vec{V}}$, and

$rot\,(\vec{r} \times \vec{V}) = \frac{\partial\vec{r}}{\partial\vec{V}} - \frac{\partial\vec{V}}{\partial\vec{r}} + \vec{r}\,div\,\vec{V} - \vec{V}\,div\,\vec{r}$.

**5.** Evaluate the divergence and the rotation of the fields;
(i) $r\,f(r)\,grad\,r + \vec{a} \times \vec{r}$
(ii) $f\,grad\,g \times g\,grad\,f$
(iii) $\vec{r} \times (\vec{a} \times \vec{r})$.

**6.** We note $\vec{u} = \vec{a} \times \vec{r}$, where $\vec{a}$ is a constant vector and $\vec{r}$ is the position vector (as usually, $u$ denotes the norm of $\vec{u}$). Find the conditions on the real functions $F$ and $G$ of a real variable, such that:
(i) $\vec{u}\,F(u)$ is irrotational
(ii) $G(u)$ is harmonic.

Hint. (i) $rot\,[\vec{u}\,F(u)\,] = \frac{dF}{du}(grad\,u + \vec{u}\,) + F(u)rot\,\vec{u}$, where

$$grad\,u = \frac{(\vec{a}\times\vec{r})\times\vec{a}}{\|\vec{a}\times\vec{r}\|}, \text{ and } rot\,\vec{u} = 2\vec{a}.$$

Finally we obtain $u\frac{dF}{du} + 2F = 0$.

(ii) $\Delta G = \frac{dG}{du}div\ grad\ u + \frac{d^2G}{du^2}(grad\ u)^2$, where $div\ grad\ u = \frac{a^2}{u}$, and $(grad\ u)^2 = a^2$. Consequently, $\Delta G = 0$ implies $\frac{1}{u}\frac{dG}{du} + \frac{d^2G}{du^2} = 0$.

**7.** Verify that the following fields are bi-scalar, and write them in the standard form $\varphi\ grad\ \psi$:

(i) $\vec{V} = (a - z)y\vec{i} + (a - z)x\vec{j} + xy\vec{k}$

(ii) $\vec{V} = grad\,f + f\,grad\,g$

(iii) $\vec{V} = \vec{r}\times(\vec{a}\times\vec{r}\,)$.

Hint. (i) $\vec{V}\,rot\,\vec{V} = 0$. We write the equation of the surfaces, which are orthogonal to the field lines $(a - z)ydx + (a - z)xdy + xydz = 0$, in the form

$$\frac{ydx + xdy}{xy} + \frac{dz}{a - z} = 0,$$

we deduce $\frac{xy}{a - z} = constant$. From the relation

$$\vec{V} = \varphi\ grad\frac{xy}{a - z}$$

we identify $\varphi = \frac{1}{(a - z)^2}$.

(ii) $rot\,\vec{V} = grad\,f\times grad\,g$, hence $\vec{V}\,rot\,\vec{V} = 0$. The surfaces orthogonal to the field lines have the equations $fe^g = constant$, hence $\vec{V} = \varphi\ grad\ (fe^g)$, where $\varphi = e^{-g}$.

(iii) Write $\vec{V} = r^2\vec{a} - (\vec{a}\,\vec{r}\,)\vec{r}$; the orthogonal surfaces are cones of vertex 0 and axis $\vec{a}$, of equation $\vec{a}\,\vec{r} = Cr$, where $C = constant$. Like before, $\vec{V} = \varphi\ grad\ \frac{\vec{a}\cdot\vec{r}}{r}$, where $\varphi = r^3$.

**8.** Let $\{\vec{e}_\rho, \vec{e}_\varphi, \vec{e}_\theta\}$ be the local base in spherical coordinates $(\rho, \varphi, \theta)$, and let $\vec{V} = \frac{\cos 2\theta}{\sin\theta}\vec{e}_\rho - 2cos\theta\ \vec{e}_\theta$ be a vector field. Show that $\vec{V}$ is bi-scalar and find the scalar fields $f$ and $g$ for which $\vec{V} = f\,grad\,g$.

Hint. $rot\,\vec{V} = -\dfrac{\cos\theta}{\rho\sin\theta}\vec{e}_\varphi$, hence $\vec{V}\,rot\,\vec{V} = 0$. Using the formula

$$d\vec{r} = \vec{e}_\rho d\rho + \rho sin\theta\,\vec{e}_\varphi d\varphi + \rho\vec{e}_\theta d\theta,$$

the equation $\vec{V}\,d\vec{r} = 0$, of the surfaces orthogonal to $\vec{V}$, becomes

$$\frac{d\rho}{\rho} = \frac{\sin 2\theta}{\cos 2\theta}d\theta,$$

This equation has solutions of the form $\rho^2 \,cos\, 2\theta = C$, i.e. $g = \rho^2 \,cos\, 2\theta$.

From $\vec{V} = f\,grad\,(\rho^2 \,cos\, 2\theta)$, where

$$grad\, g = 2\rho(cos\, 2\theta\,\vec{e}_\rho - 2sin\theta cos\theta\,\vec{e}_\theta),$$

we deduce $f = \dfrac{1}{2\rho\sin\theta}$.

**9.** Show that the field $\vec{V} = \dfrac{\vec{a}}{r^3} - \dfrac{3(\vec{a}\cdot\vec{r})}{r^5}\vec{r}$, where $\vec{a}$ is a constant vector, and $\vec{r}$ is the position vector , is harmonic, and find a scalar potential of $\vec{V}$.

Hint. $rot\,\vec{V} = 0$, $div\,\vec{V} = 0$; $\vec{V} = grad\,U$, where $U = \dfrac{\vec{a}\cdot\vec{r}}{r^3} + const.$

**10.** Determine the harmonic functions (scalar fields) which depend only on one of the spherical coordinates ρ, φ or θ.

Hint. If $U$ depends only of ρ, then the Laplace equation, $\Delta U = 0$, reduces to $\dfrac{\partial}{\partial\rho}\left(\rho^2\dfrac{\partial U}{\partial\rho}\right) = 0$.

If so, it follows that $U(\rho) = \dfrac{c_1}{\rho} + c_2$.

Similarly, $U(\varphi) = c_1\varphi + c_2$, and $U(\theta) = c_1 ln\, tg\,\dfrac{\theta}{2} + c_2$ .

**11.** Determine the field $\vec{V}: \mathbb{R}^3 \to \mathscr{T}$, for which

$$rot\,\vec{V} = (y - z)\,\vec{i} + (z - x)\,\vec{j} + (x - y)\,\vec{k} \text{ and } div\,\vec{V} = -2z - x.$$

Hint. Decompose $\vec{V} = \vec{V}_1 + \vec{V}_2$, where

$$rot\,\vec{V}_1 = 0,\quad div\,\vec{V}_1 = -2z - x,$$

and

$$rot\,\vec{V}_2 = (y - z)\,\vec{i} + (z - x)\,\vec{j} + (x - y)\,\vec{k},\quad div\,\vec{V}_2 = 0.$$

It follows that $\vec{V}_1 = grad\,\varphi$, where $\Delta\varphi = -2z - x$. Taking

$$\varphi_0 = -\frac{1}{6}x^3 - \frac{1}{3}z^3,$$

we obtain $\varphi = \varphi_0 + \Phi$, where $\Delta\Phi = 0$. Consequently,

$$\vec{V}_1 = -\frac{1}{2}x^2\vec{i} - z^2\vec{k} + grad\,\Phi.$$

On the other hand $\vec{V}_2 = \vec{V}_0 + grad\,\psi$, where $\vec{V}_0 = W_1\vec{i} + W_2\vec{j} + 0\vec{k}$ is a particular vector potential of $(y-z)\,\vec{i} + (z-x)\,\vec{j} + (x-y)\,\vec{k}$. In particular, $W_1 = \frac{z^2}{2} - xz + g(x, y)$ and $W_2 = \frac{z}{2} - yz + f(x, y)$, where $\frac{\partial f}{\partial x} - \frac{\partial g}{\partial y} = x - y$.

Taking for example $f = \frac{x^2}{2}$, $g = \frac{y^2}{2}$, we obtain

$$\vec{V}_2 = \left(\frac{z^2}{2} - xz + \frac{y^2}{2}\right)\vec{i} + \left(\frac{z^2}{2} - yz + \frac{x^2}{2}\right)\vec{j} + grad\,\psi.$$

Condition $div\,\vec{V}_2 = 0$ leads to $\Delta\psi = 2z$, which is verified by $\psi_0 = z^2$, hence $\psi = z^2 + \Psi$, where $\Delta\Psi = 0$. Since $grad\,\psi_0 = 2z\vec{k}$, we obtain

$$\vec{V}_2 = \left(\frac{z^2}{2} - xz + \frac{y^2}{2}\right)\vec{i} + \left(\frac{z^2}{2} - yz + \frac{x^2}{2}\right)\vec{j} + 2z\vec{k} + grad\,\psi.$$

The solution of the problem is

$$\vec{V} = \vec{V}_1 + \vec{V}_2 =$$

$$= \left(-\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} - xz\right)\vec{i} + \left(\frac{z^2}{2} - yz + \frac{x^2}{2}\right)\vec{j} + (2z - z^2)\vec{k} + grad\,\Xi,$$

where $\Xi = \Phi + \Psi$ is an arbitrary harmonic function.

# CHAPTER X. COMPLEX INTEGRALS

## § X.1. ELEMENTS OF CAUCHY THEORY

By its construction, the complex integral is similar to the real line integral of the second type. And yet, the properties of the complex integrals of the derivable functions are so important from both theoretical and practical point of view that they are frequently qualified as *nucleus* of the Classical Mathematical Analysis. As examples of remarkable results we mention the fundamental theorem of Algebra (D'Alembert), the unification of the integral and differential calculus (Cauchy theorems allowing the evaluation of some integrals by derivations), and the applications to the real integral calculus (including some improper integrals). This part of the Complex Analysis is known as Cauchy Theory.

We begin by introducing the complex integral in its most general sense, i.e. for arbitrary functions:

**1.1. <u>The construction of the complex integral</u>.** Let $f: D \to \mathbb{C}$ be a complex function of a complex variable ($D \subseteq \mathbb{C}$), and let $\gamma \subset D$ be a simple piece-wise smooth curve (the matter about plane curves in §VI.1 remain valid since $\mathbb{R}^2 \sim \mathbb{C}$). We note by $\varphi: I \to D$, where $I = [a, b] \subset \mathbb{R}$, and $\varphi(I) = \gamma$, a *complex parameterization* of $\gamma$; more exactly, $\varphi(t) = \alpha(t) + i\,\beta(t)$, where

$$\begin{cases} x = \alpha(t) \\ y = \beta(t) \end{cases}, \quad t \in [a,b]$$

represents the real parameterization of $\gamma$ (refresh § I.2 for more details).

A *partition* of $\gamma$ is defined as a finite set of points on $\gamma$, which is noted

$$\delta = \{z_k = \varphi(t_k) \in \gamma : k = \overline{0,n}, a < t_0 < t_1 < ... < t_n = b\},$$

where $A = \varphi(a)$ and $B = \varphi(b)$ are the *endpoints* of $\gamma$. The number

$$\nu(\delta) = \max\{|z_k - z_{k-1}| : z_k \in \delta, k = \overline{1,n}\}$$

is called *norm* of the partition $\delta$.

With each partition we associate systems of *intermediate points*, which are sets of the form (see Fig. X.1.1)

$$\mathscr{S} = \{\zeta_k = \varphi(\theta_k) \in \gamma : k = \overline{1,n}, t_{k-1} \le \theta_k \le t_k\}.$$

Finally, the numbers

$$\sigma_{f,\gamma}(\delta, \mathscr{S}) = \sum_{k=1}^{n} f(\zeta_k)(z_k - z_{k-1})$$

are called Riemann *integral sums* of the function $f$ on the curve $\gamma$, attached to the division $\delta$ and to the system $\mathscr{S}$ of intermediate points.

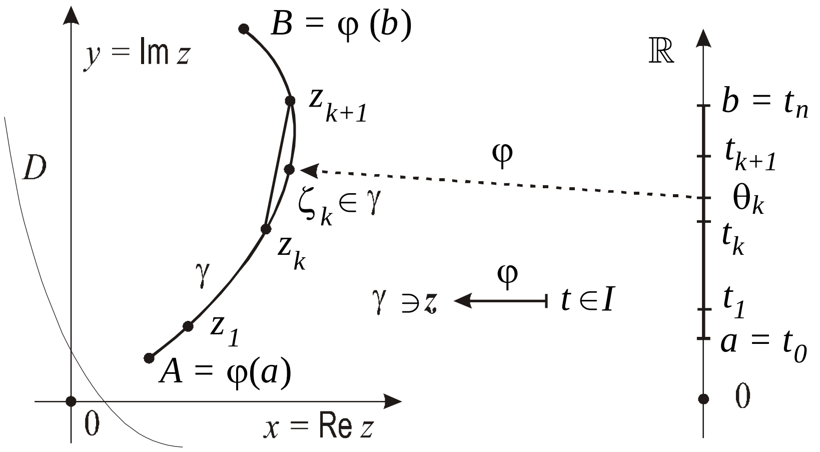



Fig.X.1.1.

**1.2. <u>Definition</u>.** We say that $f$ is *integrable* on the curve $\gamma$ if there exists the limit of the generalized sequence (net) of integral sums

$$\lim_{\nu(\delta)\to 0} \sigma_{f,\gamma}(\delta,\mathscr{S}) \in \mathbb{C}.$$

Alternatively, if we work with usual sequences of integral sums, then we ask the uniqueness of this limit for all sequences of partitions ($\delta_p$) with $\nu(\delta_p) \to 0$, and all systems of intermediate points.

If this limit exists, then we call it *integral* of $f$ on $\gamma$, and we note it

$$\int_\gamma f(z)\,dz \ .$$

The notation $\oint_\gamma f(z)\,dz$ is sometimes agreed, since the complex integral is defined on curves, by analogy to the real line integral.

The first natural question about a complex integral concerns its existence and evaluation. This is solved by the following:

**1.3. <u>Theorem</u>.** The continuous functions are integrable on piece-wise smooth curves, and their integrals reduce to real line integrals of the second type. More exactly, if $f = P + i\,Q$, then

$$\int_\gamma f(z)dz = \int_\gamma Pdx - Qdy + i\int_\gamma Qdx + Pdy \,.$$

<u>Proof</u>. If $\zeta_k = \xi_k + i\eta_k = \varphi(\theta_k)$, where $k = \overline{1,n}$, then the values of $f$ are

$$f(\zeta_k) = P(\xi_k,\eta_k) + iQ(\xi_k,\eta_k)\,.$$

Similarly, if we note $z_k = x_k + iy_k \in \delta$ for all $k = \overline{1,n}$, then

$$|z_k - z_{k-1}| = \sqrt{(x_k - x_{k-1})^2 + (y_k - y_{k-1})^2} \ ,$$

hence $\nu(\delta)$ equals the norm of $\delta$ as partition of the real curve $\gamma$ in $\mathbb{R}^2$. If we introduce these elements in the complex integral sums of $f$ on $\gamma$, then we can separate the real and imaginary parts of these sums, and we obtain:

$$\sigma_{f,\gamma}(\delta,\mathscr{S}) = \sum_{k=1}^{n}[P(\xi_k,\eta_k)(\xi_k-\xi_{k-1})-Q(\xi_k,\eta_k)(\eta_k-\eta_{k-1})] +$$

$$+\, i \sum_{k=1}^{n}[Q(\xi_k,\eta_k)(\xi_k-\xi_{k-1})+P(\xi_k,\eta_k)(\eta_k-\eta_{k-1})] \ .$$

It is easy to see that these sums converge to the real line integrals

$$\int_\gamma Pdx - Qdy = \lim_{\nu(\delta)\to 0} \sum_{k=1}^{n}[P(\xi_k,\eta_k)(\xi_k-\xi_{k-1})-Q(\xi_k,\eta_k)(\eta_k-\eta_{k-1})]$$

$$\int_\gamma Qdx + Pdy = \lim_{\nu(\delta)\to 0} \sum_{k=1}^{n}[Q(\xi_k,\eta_k)(\xi_k-\xi_{k-1})+P(\xi_k,\eta_k)(\eta_k-\eta_{k-1})].$$

These limits exist according to theorem ◇

**1.4.<u>Corollary</u>.** If $f: D\to\mathbb{C}$ is a continuous function, and $\gamma\subset D$ is a piece-wise smooth curve of parameterization $\varphi: [a, b]\to\mathbb{C}$, then

$$\int_\gamma f(z)\,dz = \int_a^b (f\circ\varphi)(t)\,\varphi^{/}(t)\,dt \ .$$

<u>Proof</u>. The complex parameterization $\varphi(t) = \alpha(t) + i\,\beta(t)$, $t\in[a,b]$, comes from the real parameterization

$$\begin{cases} x=\alpha(t) \\ y=\beta(t) \end{cases}, \quad t\in[a,b],$$

hence the hypothesis that $\gamma$ is piece-wise smooth means that the functions $\alpha$ and $\beta$ (and consequently $\varphi$ ) have continuous derivatives on a finite number of subintervals of $[a, b]$. The real line integrals from the previous theorem become definite Riemann integrals on $[a, b]$, i.e.

$$\int_\gamma Pdx - Qdy = \int_a^b \Big[P(\alpha(t),\beta(t))\,\alpha^{/}(t) - Q(\alpha(t),\beta(t))\,\beta^{/}(t)\Big]\,dt \ ,$$

$$\int_\gamma Qdx + Pdy = \int_a^b \Big[Q(\alpha(t),\beta(t))\,\alpha^{/}(t) + P(\alpha(t),\beta(t))\,\beta^{/}(t)\Big]\,dt \ .$$

To accomplish the proof, we replace $f = P + i\,Q$, and $\varphi^{/} = \alpha^{/} + i\,\beta^{/}$ in these integrals, and restrain the result in a complex form. ◇

**1.5. <u>Example</u>.** The function $f: \mathbb{C}\setminus\{z_0\}\to\mathbb{C}$ , of values $f(z)=(z-z_0)^{-1}$, is integrable on the circle $C(z_0, r)$, centered at $z_0$, of radius $r > 0$, and

$$\int_{C(z_0,r)} \frac{dz}{z-z_0} = 2\pi i \ .$$

In fact, the integral exists because $f$ is continuous on $D = \mathbb{C}\setminus\{z_0\}$, hence also on $C(z_0, r)$, and the circle (traced once) is a simple smooth curve in $D$. Using the complex parameterization of this circle

$$\varphi(t) = z_0 + r\,e^{it}, \ t\in[0, 2\pi] \ ,$$

we obtain $\varphi'(t) = r\,i\,e^{it}$, and $(f\circ\varphi)(t) = \dfrac{1}{r\,e^{it}}$. Consequently, according to the formula from Corollary 1.4, the value of the integral is

$$\int_{C(z_0,r)} \frac{dz}{z-z_0} = \int_0^{2\pi} i\ dt = 2\pi\, i\ .$$

The general properties of the complex integrals correspond to the similar properties of the real line integrals of the second type:

**1.6. __Theorem__.** The following relations hold for continuous functions on piece-wise smooth curves:

($i$) $\int_\gamma (\alpha f+\beta g)(z)dz = \alpha\int_\gamma f(z)dz + \beta\int_\gamma g(z)dz\,, \forall\alpha,\beta\in\mathbb{C},\ \forall f,g\in C^0_{\mathbb{C}}(\gamma)$

(known as *linearity* relative to the function);

($ii$) $\int_{\gamma_1\cup\gamma_2} f(z)\,dz\ =\ \int_{\gamma_1} f(z)\,dz\ +\ \int_{\gamma_2} f(z)\,dz\,,\ \forall f\in C^0_{\mathbb{C}}(\gamma_1\cup\gamma_2)$

(called *additivity* relative to the concatenation of the curves);

($iii$) $\int_{\gamma^-} f(z)dz = -\int_\gamma f(z)dz\,,\ \forall f\in C^0_{\mathbb{C}}(\gamma)$, where $\gamma$ and $\gamma^-$ are contrarily traced (named *orientation* relative to the sense on $\gamma$).

__Proof__. Without going into details, we recognize here the similar properties of the real line integrals of the second type, hence it is enough to recall the connection established in Theorem 1.3. ◇

The following property of boundedness reminds of real line integrals of the first type, since it involves the *length* of a curve.

**1.7. __Theorem__.** (Boundedness of the complex integral) Let $f$ and $\gamma$ be as in the construction 1.1. If $M = \sup\limits_{z\in\gamma}|f(z)|$ , and $L$ is the length of $\gamma$, then

$$\left|\int_\gamma f(z)dz\right| \le M\cdot L\ .$$

__Proof__. Because I = [a, b] is a compact set in $\mathbb{R}$, and the parameterization $\varphi$ is continuous, it follows that $\gamma\ =\varphi\,(I)$ is a compact set in $\mathbb{C}$. The continuity of $|f|$ assures the existence of $M<\infty$, such that $|f(\zeta)|\le M$ at all $\zeta\in\gamma$. Since the smooth curves are rectifiable (i.e. they have *length*), there exists

$$L \overset{def.}{=} \sup_\delta\{\sum_{k=1}^n |z_k - z_{k-1}| : z_k\in\delta\} < \infty\,.$$

Consequently, for the modulus of the integral sums, we obtain

$$|\sigma_{f,\gamma}(\delta,\mathscr{S})| \le \sum_{k=1}^n |f(\zeta_k)|\cdot|z_k - z_{k-1}| \le M\sum_{k=1}^n |z_k - z_{k-1}| \le ML\,,$$

where it is enough to take the limit $\nu(\delta)\to 0$. ◇

**1.8. <u>Corollary</u>.** Let $D$ be a domain in $\mathbb{C}$, and let $\gamma \subset D$ be a piece-wise smooth curve. For each $n \in \mathbb{N}$, we define a function $f_n : D \to \mathbb{C}$, which is continuous on $\gamma$. If the sequence $(f_n)$ is uniformly convergent on $\gamma$, then

$$\lim_{n\to\infty} \int_\gamma f_n(z)dz = \int_\gamma \lim_{n\to\infty} f_n(z)dz .$$

<u>Proof</u>. By hypothesis, $f \overset{u.}{\underset{\gamma}{=}} \lim\limits_{n\to\infty} f_n$ means that for each $\varepsilon > 0$ there exists $n_0(\varepsilon) \in \mathbb{N}$, such that $n > n_0(\varepsilon)$ implies

$$M \overset{not.}{=} \sup_{z\in\gamma} |f_n(z) - f(z)| < \frac{\varepsilon}{L} ,$$

where $L$ stands for the length of $\gamma$. According to theorem 1.7, we obtain

$$\left| \int_\gamma f_n(z)dz - \int_\gamma f(z)dz \right| \le \varepsilon ,$$

which is a reformulation of the claimed equality. ◇

**1.9. <u>Corollary</u>.** Let function $f$ be analytically defined by

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n .$$

If $\gamma$ is a piece-wise smooth curve in the disk of convergence of this power series, then we may integrate term by term, i.e.

$$\int_\gamma f(z)dz = \sum_{n=0}^{\infty} a_n \int_\gamma (z - z_0)^n dz .$$

<u>Proof</u>. The partial sums of the given power series can play the role of $f_n$ in the previous Corollary. ◇

**1.10. <u>Remark</u>.** An important property of the real line integrals of the second type concerns the independence on curve. In § VI.3, we have seen that this is the case of conservative fields, which derive from a potential. Simple examples (see the problems at the end, as well as $I \neq 0$ in Example 1.5, etc.) show that the complex integral generally depends on the curve of integration. However, if the integrated function is $\mathbb{C}$-derivable, then its integral does not depend on curve, but only of its endpoints. The following theorem states conditions for this case, which will be assumed in the entire forthcoming theory:

**1.11. <u>Cauchy's Fundamental Theorem</u>.** Let $f: D \to \mathbb{C}$, where $D \subseteq \mathbb{C}$, be a $\mathbb{C}$-derivable function, and let $\gamma \subset D$ be a simple, closed, piece-wise smooth curve. If the interior of $\gamma$ is included in $D$, i.e. $(\gamma) \subset D$, then

$$\int_\gamma f(z)\,dz = 0 .$$

Proof (based on the additional hypothesis that $P = \mathrm{Re} f, Q = \mathrm{Im} f \in C^1_{\mathbb{R}}(D)$).

The integrability of $f$ on $\gamma$ is assured by Theorem 1.3. The additional hypothesis allows us to use the Green's formula from § VII.2, which gives

$$\int_\gamma Pdx - Qdy = -\iint_{(\gamma)} \left[\frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y}\right] dxdy$$

$$\int_\gamma Qdx + Pdy = \iint_{(\gamma)} \left[\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}\right] dxdy.$$

Because $f$ is derivable on $D$, hence also on ( $\gamma$), it follows that the Cauchy-Riemann conditions hold, hence the double integrals from above vanish. It remains to use Theorem 1.3. ◇

**1.12. Remark.** The assertion of the above theorem is correct without additional hypotheses, but the proof becomes much more complicated (see for example [CG], [G-S], [H-M-N], [MI], etc.).

Before discussing more consistent consequences of the above Theorem 1.11, we mention several immediate corollaries, which are also significant for the relation between complex and real integrals. These properties hold on domains of a particular form:

**1.13. Definition.** We say that a domain $D \subseteq \mathbb{C}$ is *simply connected* if the interior of every closed curve from $D$ is also in $D$, i.e.

$$\gamma \subset D \Rightarrow (\gamma) \subset D.$$

In the contrary case, when there exist curves $\gamma \subset D$ for which $(\gamma) \not\subset D$, we say that $D$ is *multiply connected* (anyway, $D$ is connected, since *domain* means *open* and *connected*). Here we avoid further considerations on the *order* of multiplicity (based on *homotopic* curves), and other properties of the domains, but the interested reader may consult [BN], [G-S], [LS], etc.

**1.14. Examples.** (a) The following sets are simply connected:

- $\mathbb{C}$, $\overline{\mathbb{C}}$, $\{z \in \mathbb{C} : \mathrm{Re} z \geq 0\}$, and other half-planes;
- Disks (i.e. interior of circles), and interior of simple closed curves;
- Arbitrary intersections of simply connected sets.

(b) Most frequently, the multiply connected sets have the form:

- $\mathbb{C} \setminus \{z_0\}$, $\mathbb{C} \setminus F$, where $F \subset \mathbb{C}$ is finite, $\mathbb{C} \setminus \mathbb{N}$, etc.
- $D(z_0, r) \setminus \{z_0\}$, i.e. disks without center, $D(z_0, r) \setminus F$, where $F$ is a finite (or even infinite) set of "missing" points;
- Connected sets with "missing" points or "missing" sub-domains.

**1.15. Corollary.** If $D \subseteq \mathbb{C}$ is a simply connected domain, and $f: D \to \mathbb{C}$ is a derivable function, then:

*(i)* The integral of $f$ does not depend on the curve;
*(ii)* $f$ has primitives on $D$;
*(iii)* The Leibniz-Newton formula holds for the integral of $f$.

Proof. ($i$) Let $\gamma_1$ and $\gamma_2$ be two curves in $D$, which have the same endpoints, say $A$ and $B$. The curve $\gamma = \gamma_1 \cup \gamma_2^-$, obtained by concatenation, is closed, and since $D$ is simply connected, we have $(\gamma) \subset D$. According to Theorem 1.11, it follows that $\int_\gamma f(z)\,dz = 0$, hence by virtue of properties ($ii$-$iii$) from Theorem 1.6, we obtain $\int_{\gamma_1} f(z)\,dz - \int_{\gamma_2} f(z)\,dz = 0$.

($ii$) We fix $z_0 \in D$, and we prove that the function $F : D \to \mathbb{C}$, of values

$$F(z) = \int_{z_0}^{z} f(\zeta)\,d\zeta$$

is a primitive of $f$, i.e. $F$ is derivable on $D$ and $F' = f$. In fact, for each $z_0 \in D$, there exists $\delta_0(z) > 0$, small enough to assure the implication

$$|\Delta z| < \delta_0(z) \Rightarrow z + \Delta z \in D.$$

Because $\int_z^{z+\Delta z} d\zeta = \Delta z$ holds at all $z$ and $\Delta z$ in $\mathbb{C}$, we may write

$$E(z, \Delta z) \overset{not.}{=} \left| \frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) \right| =$$

$$= \left| \frac{1}{\Delta z} \int_z^{z+\Delta z} f(\zeta)\,d\zeta - \frac{1}{\Delta z} \int_z^{z+\Delta z} f(z)\,d\zeta \right| = \frac{1}{|\Delta z|} \left| \int_z^{z+\Delta z} [f(\zeta) - f(z)]\,d\zeta \right| .$$

Using the independence on curve of the last integral, we may evaluate it on the straight-line segment $[z, z + \Delta z]$. For this integral, the Property 1.7, of boundedness, holds with $L = |\Delta z|$, hence

$$E(z, \Delta z) \le M \overset{not.}{=} \max\{ |f(\zeta) - f(z)| : \zeta \in [z, z+\Delta z] \} .$$

The derivability of $f$ implies its continuity, hence for each $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$, such that $|\Delta z| < \min\{\delta(\varepsilon), \delta_0(z)\}$ implies $|f(z+\Delta z) - f(z)| < \varepsilon$. In this situation, a fortiori $|f(\zeta) - f(z)| < \varepsilon$, hence

$$f(z) = \lim_{\Delta z \to 0} \frac{F(z+\Delta z) - F(z)}{\Delta z} \overset{not.}{=} F'(z).$$

($iii$) We have to show that the formula

$$\int_{z_1}^{z_2} f(z)dz = G(z_2) - G(z_1)$$

holds for arbitrary $z_1, z_2 \in D$, and for arbitrary primitive $G$ of $f$. In fact, the previous property ($ii$) points out a particular primitive of $f$, namely

$$F(z) = \int_{z_1}^{z} f(\zeta)\,d\zeta .$$

Because the difference of two primitives of the same function is a constant, i.e. $(F - G)' = 0$ implies the existence of $C \in \mathbb{C}$, we have $F(z) - G(z) = C$ at all $z \in D$. In particular, we take here $z = z_1$ and $z = z_2$. ◇

We start the series of major consequences of the fundamental theorem by the case of a multiply connected domain:

**1.16. <u>Theorem</u>.** Let $D \subseteq \mathbb{C}$ be a domain, and let $\gamma$, $\gamma_1$, ..., $\gamma_n$ be pair-wise disjoint closed (and, as usually, simple and piece-wise smooth) curves in $D$. If $f: D \to \mathbb{C}$ is a derivable function on $D$, and $(\gamma) \setminus \left[ \bigcup_{k=1}^{n} (\gamma_k) \right] \subset D$, then

$$\int_\gamma f(z)\,dz = \sum_{k=1}^{n} \int_{\gamma_k} f(z)\,dz .$$

<u>Proof</u>. The idea is to apply the fundamental theorem 1.11 to $f$ and some closed curve $\Gamma$, for which $(\Gamma) \subset D$. To make it possible, we take $A_k \in \gamma$ and $B_k \in \gamma_k$ for each $k = \overline{1,n}$, e.g. the closest points between $\gamma$ and $\gamma_k$, and we connect them by straight-line segments (as in Fig. X.1.2). The sought for curve $\Gamma$ results by the following concatenation:

$$\Gamma = \gamma\Big|_{A_n A_1} \cup [A_1, B_1] \cup \gamma_1^- \cup [B_1, A_1] \cup \gamma\Big|_{A_1 A_2} \cup \ldots \cup [B_n, A_n] .$$

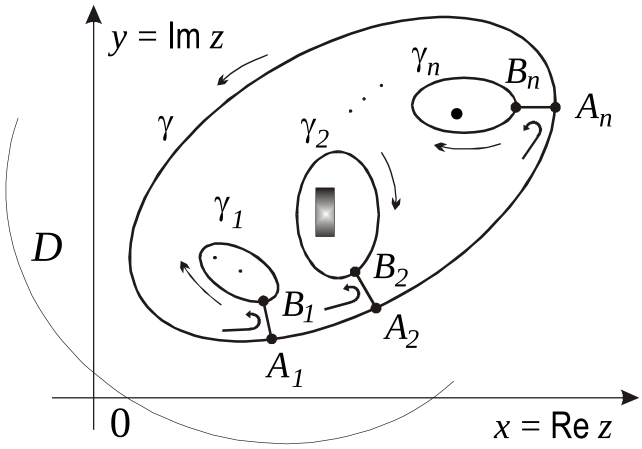



Fig. X.1.2.

Consequently, according to 1.6 (*ii*), we can decompose the integral on $\Gamma$ in a sum of integrals on the constituent arcs. Because the segments $[A_k, B_k]$ and $[B_k, A_k]$ are opposite in order, we have

$$\int_{[A_k, B_k]} f(z)\,dz + \int_{[B_k, A_k]} f(z)\,dz = 0 .$$

In addition $\gamma\Big|_{A_n A_1} \cup \gamma\Big|_{A_1 A_2} \cup \ldots \cup \gamma\Big|_{A_{n-1} A_n} = \gamma$, hence

$$\int_{\gamma|_{A_n A_1}} f(z)\,dz + \int_{\gamma|_{A_1 A_2}} f(z)\,dz + \ldots + \int_{\gamma|_{A_{n-1} A_n}} f(z)\,dz = \int_\gamma f(z)\,dz .$$

To complete the proof, we apply the property 1.6 (*iii*) on $\gamma_k^-$, $\forall k = \overline{1,n}$, and the fundamental theorem on $\Gamma$, i.e. $\int_\Gamma f(z)\,dz = 0$. ◇

**1.17. Remarks.** The main applications of the above theorem consist in reducing the integral on $\gamma$ to integrals on $\gamma_k$ , which in general are simpler. In particular, if $n$ =1, then $\int_\gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz$, whenever $(\gamma)\setminus(\gamma_1)\subset D$. This property is frequently formulated in terms of continuous deformation of $\gamma$ to $\gamma_1$ , realized inside $D$. For example, using 1.5, we obtain that

$$\int_\gamma \frac{dz}{z-z_0} = 2\pi\, i$$

holds for arbitrary curve $\gamma$ under the condition $z_0 \in (\gamma)$. To complete the list of values of this integral, we mention that it vanishes if $z_0 \notin \overline{(\gamma)}$, i.e. $z_0$ is exterior to $\gamma$, and the case $z_0 \in \gamma$ is undecided (see the next section).

The next theorem, by Cauchy too, states a remarkable relation between integrals and derivatives:

**1.18. Theorem (Cauchy formulas for derivable functions).** Let $f: D\to\mathbb{C}$ be a derivable function on the domain $D\subseteq\mathbb{C}$. If $\gamma$ is a closed (simple and piece-wise smooth) curve in $D$, such that $(\gamma)\subset D$, then the formula

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

holds at every $z_0 \in (\gamma)$, and for arbitrary $n\in\mathbb{N}$ .

Proof. Case $n$ = 0. We have to show that

$$\int_\gamma \frac{f(z)}{z-z_0}\,dz = f(z_0)\cdot 2\pi i \ .$$

Using the result in Example 1.5, and Remark 1.17, this relation becomes

$$\int_\gamma \frac{f(z)}{z-z_0}\,dz = f(z_0)\cdot\int_\gamma \frac{dz}{z-z_0} \ .$$

Because $f$ is derivable at $z_0$, there exists $M > 0$ such that the inequality

$$\left|\frac{f(z)-f(z_0)}{z-z_0}\right| < M$$

holds at all $z$ in a neighborhood of $z_0$ . If we replace $\gamma = C(z_0 , r)$, then according to Theorem 1.7, we have

$$\left|\int_{C(z_0,r)} \frac{f(z)-f(z_0)}{z-z_0}\,dz\right| \le M\cdot 2\pi\, r .$$

It remains to take $r\to 0$.

The remaining cases involve mathematical induction and will be omitted (the interested reader may consult [CG], [G-S], etc.). We mention that an important part of the proof concerns the implicit assertion of the theorem, namely the existence of all higher order derivatives at each point of the domain where $f$ is once derivable. ◇

**1.19. <u>Corollary</u>.** If the function $f: D \to \mathbb{C}$ is derivable on the domain $D \subseteq \mathbb{C}$, then it is infinitely derivable on $D$, i.e. there exist $f^{(n)}(z_0)$ at each $z_0 \in D$, and for all $n \in \mathbb{N}$.

**1.20. <u>Remarks</u>.** We may use the Cauchy formulas for derivable functions to evaluate complex integrals by the simpler way of derivation. The case in 1.5 is immediately recovered from Theorem 1.18, applied to the identically constant function $f: \mathbb{C} \to \{1\}$; no derivation is necessary. The same theorem, applied to the same function, leads to

$$\int_{C(z_0,r)} \frac{dz}{(z-z_0)^{n+1}} = 0$$

at all $z_0 \in \mathbb{C}$, and for all $r > 0$, $n \in \mathbb{N}^*$.

A lot of complex integrals allow the form of the Cauchy formulas, hence we can calculate them by derivations. For example,

$$\int_{C(i,1)} \frac{dz}{(z^2+1)^2} = \int_{C(i,1)} \frac{\frac{1}{(z+i)^2}}{(z-i)^2} dz = 2\pi i \cdot \left[ \frac{1}{(z+i)^2} \right]'_{z=i}$$

follows for $z_0 = i$, $n = 1$, and $f(z) = (z+i)^{-2}$.

The Cauchy formulas have important theoretical consequences:

**1.21. <u>Theorem</u> (bounding the derivatives).** Let $f: D \to \mathbb{C}$ be a derivable function on the domain $D \subseteq \mathbb{C}$, and let $z_0 \in D$. If $\gamma = C(z_0, r)$, and $r$ is small enough to ensure the inclusion $(\gamma) \subset D$, then

$$\left| f^{(n)}(z_0) \right| \le \frac{n!\, M(z_0,r)}{r^n} ,$$

where $M(z_0,r) = \sup\{|f(z)| : z \in C(z_0,r)\}$, and $n \in \mathbb{N}$.

<u>Proof</u>. If we apply Theorem 1.7 to the Cauchy formula for $f^{(n)}$, then we get

$$\left| f^{(n)}(z_0) \right| \le \frac{n!}{2\pi} \cdot L \cdot \sup\left\{ \frac{|f(z)|}{|z-z_0|^{n+1}} : z \in C(z_0,r) \right\},$$

where $L = 2\pi r$ is the length of $\gamma$. ◇

**1.22. <u>Theorem</u> (Liouville).** If a function is derivable on the entire complex plane, and bounded, then it is necessarily constant.

<u>Proof</u>. Using the hypothesis of boundedness, we may note $M = \sup\limits_{z \in \mathbb{C}} |f(z)|$, such that the inequality $M(z_0,r) \le M$ holds for arbitrary $z_0 \in \mathbb{C}$, and $r > 0$. According to the previous theorem, written for $n = 1$, we have

$$\left| f'(z_0) \right| \le \frac{M}{r} \underset{r \to \infty}{\to} 0 .$$

Consequently $f' = 0$ on $\mathbb{C}$, hence $f$ reduces to a constant. ◇

Finally, a remarkable consequence of the *Cauchy Theory* is the following property of $\mathbb{C}$ of being *algebraically closed*, which is considered to be *the fundamental theorem of the Algebra*:

**1.23. <u>Theorem</u> (D'Alembert).** Every polynomial $P_n$, of degree $n \geq 1$, with coefficients from $\mathbb{C}$, has at least one root in $\mathbb{C}$.

<u>Proof</u>. In the contrary case, when $P_n$ never vanishes on $\mathbb{C}$, we can define a function $f: \mathbb{C} \to \mathbb{C}$, which takes the values $f(z) = 1/P_n(z)$. According to the algebraic properties of the derivable functions (discussed in Chapter IV), it follows that $f$ is derivable on $\mathbb{C}$. In addition, $f$ is bounded. In fact, because $\lim\limits_{z \to \infty} P_n(z) = \infty$, we have $\lim\limits_{z \leftarrow \infty} f(z) = 0$, hence there exists some $r > 0$, such that $|f(z)| \leq 1$ whenever $|z| > r$. The boundedness of a continuous function on compact sets guarantees the existence of a number $M_r > 0$, such that

$$\left(|z| \leq r\right) \Rightarrow \left(|f(z)| \leq M_r\right).$$

Since $f$ is derivable and bounded on $\mathbb{C}$, the Liouville's Theorem says that $f$ is constant, which is not the case if $n \geq 1$. ◇

The list of consequences of the Cauchy's fundamental theorem continues with many other remarkable results, including those from the next section. Without going into details, we enounce here an extension of the Corollaries 1.8 and 1.9, which shows that the Cauchy Formulas "resist" to a limiting process:

**1.24. <u>Theorem</u> (Weierstrass).** Let $D \subseteq \mathbb{C}$ be a domain. If:

(*i*) $\gamma$ is a closed (simple and piece-wise smooth) curve in $D$, and $(\gamma) \subset D$;

(*ii*) $f_n : D \to \mathbb{C}$ are derivable on $D$, $\forall n \in \mathbb{N}$;

(iii) there exists $F \overset{u.}{\underset{\gamma}{=}} \lim\limits_{n \to \infty} f_n$;

then there exists $\varphi \overset{a.u.}{\underset{(\gamma)}{=}} \lim\limits_{n \to \infty} f_n$, such that

(*a*) $\varphi : (\gamma) \to \mathbb{C}$ is derivable;

(*b*) $\varphi^{(k)} \overset{a.u.}{\underset{(\gamma)}{=}} \lim\limits_{n \to \infty} f_n^{(k)}$, $\forall k \in \mathbb{N}$; and

(*c*) $\varphi^{(k)}(z) = \dfrac{k!}{2\pi i} \displaystyle\int_\gamma \frac{F(\zeta)}{(\zeta - z)^{k+1}} d\zeta$, $\forall z \in (\gamma)$, and $\forall k \in \mathbb{N}$.

In particular, we may apply this theorem to series.

## PROBLEMS §X.1.

**1.** Evaluate the integrals on [0, 1] of the following complex functions:

(a) $f(t) = \dfrac{t+i}{t-i}$, (b) $g(t) = e^{i\omega t}$, (c) $h(t) = \sin(i+t)$, (d) $\operatorname{sign}(i\,t) = \dfrac{i\,t}{|i\,t|}$.

Hint. Identify the real and imaginary parts of the given functions, and integrate them separately. In the example (a), from $f(t) = \dfrac{t^2-1}{t^2+1} + i\,\dfrac{2t}{t^2+1}$ , we obtain

$$\int_0^1 f(t)\,dt = \int_0^1 \frac{t^2-1}{t^2+1}dt + i\int_0^1 \frac{2t}{t^2+1}dt \;=\; \left(t - 2arctgt\right)\Big|_0^1 + i\ln(t^2+1)\Big|_0^1 .$$

**2.** Evaluate the complex integral $I(\gamma) = \int_\gamma |z|\,dz$ along the following curves:

(a) Straight-line segment $\gamma = [-i, i]$ ;
(b) Left-hand half-circle centered at 0, of radius 1 ;
(c) Broken line $\gamma = [-i, 1] \cup [1, i]$.

Hint. The integral refers to the real function $|z|$, but the variable is complex. Replace $z$ and $dz$ from the parameterization of the curve, according to the formulas in 1.3 and 1.4.

**3.** Study whether $\int_\gamma (i+\bar{z})\,dz$ depends on $\gamma$ or not, where $\gamma$ is a curve of endpoints $z_0 = 0$ and $z_1 = 1 + i$ .

Hint. According to theorem 1.11, since the function $i+\bar{z}$ is not derivable, there are chances the integral to depend on the curve. To point out this dependence, evaluate the integral on the straight-line segment $[0, 1+i]$, on the broken line $[0, 1] \cup [1, 1+i]$, and on arcs of parabola, circle, etc.

**4.** Let $\gamma$ be a closed (simple and piece-wise smooth) curve in $\mathbb{C}$. Show that the area of $(\gamma)$ has the expression

$$A = \frac{1}{2i}\int_\gamma \bar{z}\,dz \;.$$

Hint. If we note $z = x + iy$, then Theorem 1.3 leads to real line integrals

$$\int_\gamma \bar{z}\,dz = \int_\gamma x\,dx + y\,dy + i\int_\gamma -y\,dx + x\,dy\,.$$

The real part of this expression vanishes as an integral of a total differential

$$x\,dx + y\,dy = \tfrac{1}{2}d(x^2+y^2)\,.$$

According to Proposition VI.3.15, the imaginary part equals $2A$ .

**5.** Some of the following complex integrals can be calculated by Leibniz-Newton formulas. Identify them, and find their values.

$I_1=\int_{-1}^{1} e^z\,dz$; $I_2=\int_{-i}^{i} e^z \sin z\,dz$; $I_3=\int_{1}^{i} z\cos\pi z^2 dz$; $I_4=\int_{-1}^{1}\frac{dz}{z}$; $I_5=\int\limits_{C(0,1)}\sqrt{z}\,dz$.

Hint. The method is working for $I_1$ (as in $\mathbb{R}$), $I_2$ (by parts), and $I_3$ (changing $z^2=\zeta$). In $I_4$, the domain $\mathbb{C}\setminus\{0\}$ is not simply connected, and in $I_5$ we have a multi-valued function. However, there exist convenient cuts.

**6.** Evaluate $I(r)=\int_{C(0,r)}\frac{dz}{z^2+1}$, where $0<r\neq 1$.

Hint. If $r\in(0,1)$, then $I(r)=0$ according to the fundamental theorem 1.11. If $r>0$, then using theorem 1.16 with convenient $r_1$, $r_2>0$, we obtain

$$I(r)=\int_{C_1(i,r_1)}\frac{dz}{z^2+1}+\int_{C_2(-i,r_2)}\frac{dz}{z^2+1}.$$

Taking into account the example 1.5, and the decomposition

$$\frac{1}{z^2+1}=\frac{1}{2i}\frac{1}{z-i}-\frac{1}{2i}\frac{1}{z+i},$$

we find $I(r)=0$ again.

**7.** Using the Cauchy formulas, evaluate the integrals:

$$I_1=\int\limits_{|z-1+i|=\sqrt{2}}\frac{dz}{z^5-2z^3+2z^2-3z+2};\quad I_2=\int\limits_{C(0,1)}\frac{dz}{z^5+2z^3};$$

$$I_3=\int\limits_{C(0,1)}\frac{1}{\sin z}dz;\quad I_4=\int\limits_{C(0,1)}\frac{e^z-1}{z\sin z}dz;\quad I_5=\int\limits_{|z-1|=1}\frac{\sin\pi z}{(z^2-1)^2}dz.$$

Hint. In $I_1$, use the Horner schema to find the roots of the denominator. In $I_2$, the Cauchy formula holds with $z_0=0$, and $n=2$. In $I_3$, put forward the function $\frac{z}{\sin z}$. In $I_4$, we have $\lim\limits_{z\to 0}\frac{e^z-1}{\sin z}=1$, and in $I_5$, $\lim\limits_{z\to 1}\frac{\sin\pi z}{z-1}=-1$.

**8.** Evaluate $\int\limits_{3|z|=2}\frac{F(\zeta)}{(2\zeta-1)^2}d\zeta$, where $F(\zeta)=\sum\limits_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\zeta^{2k+1}$.

Hint. The power series of $F$ is convergent in the unit disk, where we can use the Weierstrass theorem. Take into account that $F'(z)=\frac{1}{1+z^2}$.

## § X.2. RESIDUES

This section is a further development of the idea of reducing the integrals to derivatives via special (namely Laurent) power series. The theoretic basis follows from the Cauchy theory, which has been sketched in the previous section. From a practical point of view, the interest is to gain new powerful tools for the calculus of complex as well as real integrals.

**2.1. <u>Theorem</u> (Laurent).** Let $D \subseteq \mathbb{C}$ be a domain, and let $z_0 \in \mathbb{C}$ be a point such that $\Theta(z_0, r, R) \subset D$ for some $r, R \in \mathbb{R}_+$, $r < R$, where

$$\Theta(z_0, r, R) \overset{not.}{=} \{z \in \mathbb{C} : r < |z - z_0| < R\}$$

represents the crown centered at $z_0$, of radiuses $r$ and $R$ (as in Fig.X.2.1). If $f : D \to \mathbb{C}$ is derivable on $D$, then, at each $z \in \Theta(z_0, r, R)$, its value is

$$f(z) = \ldots + \frac{a_{-p}}{(z - z_0)^p} + \ldots + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \ldots + a_n(z - z_0)^n + \ldots$$

<u>Proof</u>. We express $f(z)$ by the Cauchy formula, using the closed curve

$$\Gamma = C_R \cup [A, B] \cup C_r^- \cup [B, A] \ .$$

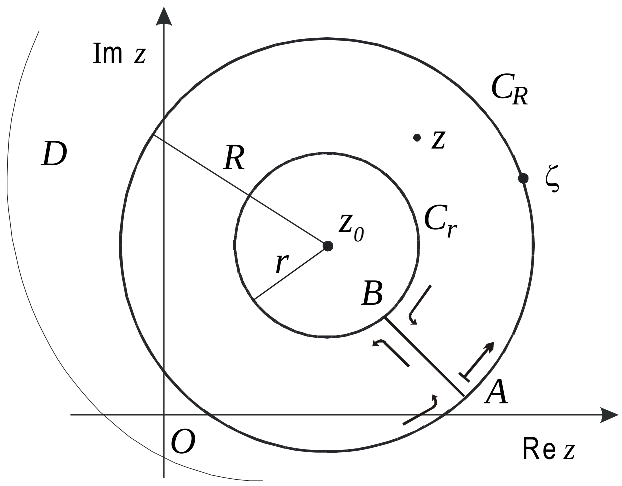



Fig.X.2.1.

In fact, $z \in (\Gamma) \subset \Theta(z_0, r, R) \subset D$, hence Theorem X.1.18 gives

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z} d\zeta \ .$$

Because $\int_\Gamma = \int_{C_R} + \int_{[A,B]} + \int_{C_r^-} + \int_{[B,A]}$ and $\int_{[A,B]} + \int_{[B,A]} = 0$, we obtain

$$f(z) = \frac{1}{2\pi i} \int_{C_R} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \int_{C_r} \frac{f(\zeta)}{\zeta - z} d\zeta \ .$$

In the last integrals, $\zeta$ has different positions relative to $z_0$ and $z$, namely:

<u>Case I</u>. $\zeta \in C_R$, hence $|z-z_0|<|\zeta-z_0|$. Consequently, in $\int_{C_R}$ we have

$$\frac{1}{\zeta-z}=\frac{1}{\zeta-z_0-(z-z_0)}=\frac{1}{\zeta-z_0}\cdot\frac{1}{1-\dfrac{z-z_0}{\zeta-z_0}}.$$

Because $\left|\dfrac{z-z_0}{\zeta-z_0}\right|<1$, the last fraction is the sum of a geometric series, i.e.

$$\frac{1}{1-\dfrac{z-z_0}{\zeta-z_0}}=1+\frac{z-z_0}{\zeta-z_0}+\left(\frac{z-z_0}{\zeta-z_0}\right)^2+\ldots+\left(\frac{z-z_0}{\zeta-z_0}\right)^n+\ldots$$

holds for all $\zeta \in C_R$, in the sense of the uniform convergence. If we put

$$\frac{1}{\zeta-z}=\frac{1}{\zeta-z_0}+\frac{z-z_0}{(\zeta-z_0)^2}+\ldots+\frac{(z-z_0)^n}{(\zeta-z_0)^{n+1}}+\ldots$$

in $\int_{C_R}$, then we may integrate term by term, and so we obtain

$$\frac{1}{2\pi i}\int_{C_R}\frac{f(\zeta)}{\zeta-z}d\zeta=\sum_{n=0}^{\infty}a_n(z-z_0)^n,$$

where

$$a_n=\frac{1}{2\pi i}\int_{C_R}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}d\zeta.$$

We claim that this series (of positive powers) is a.u. convergent relative to $z\in\Theta(z_0,r,R)$. In fact, the remainder of order $n$ equals

$$\mathfrak{R}_n\overset{def.}{=}\sum_{k=n+1}^{\infty}a_k(z-z_0)^k=\frac{1}{2\pi i}\sum_{k=n+1}^{\infty}(z-z_0)^k\int_{C_R}\frac{f(\zeta)}{(\zeta-z_0)^{k+1}}d\zeta.$$

Using the a.u. convergence of the geometric series on $\Theta(z_0,r,R)$, we find

$$\mathfrak{R}_n=\frac{1}{2\pi i}\int_{C_R}\frac{(z-z_0)^{n+1}f(\zeta)}{(\zeta-z_0)^{n+2}}\left(\sum_{k=0}^{\infty}\left(\frac{z-z_0}{\zeta-z_0}\right)^k\right)d\zeta=$$

$$=\frac{1}{2\pi i}\int_{C_R}\frac{f(\zeta)}{\zeta-z}\left(\frac{z-z_0}{\zeta-z_0}\right)^{n+1}d\zeta.$$

If we note $|\zeta-z_0|=\rho$, then we have $|\zeta-z_0|\ge R-\rho$. According to Theorem X.1.7, the following inequality holds

$$|\mathfrak{R}_n|\le\frac{1}{2\pi}\frac{M}{R-\rho}\left(\frac{\rho}{R}\right)^{n+1}\cdot 2\pi R,$$

where $M=\sup\{|f(\zeta)|:\zeta\in C_R\}$. From $\rho<R$ we deduce $\lim\limits_{n\to\infty}\mathfrak{R}_n=0$.

Case II. $\zeta \in C_r$, hence $|\zeta - z_0| < |z - z_0|$. In this case, in $\int_{C_r}$ we write

$$\frac{-1}{\zeta - z} = \frac{1}{z - z_0 - (\zeta - z_0)} = \frac{1}{z - z_0} \cdot \frac{1}{1 - \dfrac{\zeta - z_0}{z - z_0}} .$$

Considerations similar to those from Case I lead to the development

$$\frac{-1}{2\pi i}\int_{C_r} \frac{f(\zeta)}{\zeta - z} d\zeta = \sum_{p=1}^{\infty} \frac{a_{-p}}{(z - z_0)^p} ,$$

where

$$a_{-p} = \frac{1}{2\pi i}\int_{C_r} f(\zeta) \cdot (\zeta - z_0)^{p-1} d\zeta .$$

A similar evaluation of the remainder

$$\mathfrak{r}_p \overset{def.}{=} \sum_{k=p+1}^{\infty} \frac{a_{-k}}{(z - z_0)^k}$$

leads to the conclusion that $\lim\limits_{n \to \infty} \mathfrak{r}_p = 0$.

Combining the two cases, we obtain the proof of the claimed equality

$$f(z) = \sum_{p=1}^{\infty} \frac{a_{-p}}{(z - z_0)^p} + \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

in the sense of the a.u. convergence of each series on $\Theta(z_0, r, R)$. ◇

**2.2. <u>Remarks</u>.** We usually refer to the series

$$\ldots + \frac{a_{-p}}{(z - z_0)^p} + \ldots + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \ldots + a_n(z - z_0)^n + \ldots$$

introduced by Theorem 2.1, as a *Laurent series*. This series has two *entries*, and its convergence means concomitant convergence of the two series,

$\sum\limits_{n=0}^{\infty} a_n (z - z_0)^n$, called *regular part* (*Taylor*, of *positive powers*, etc.),

and $\sum\limits_{p=1}^{\infty} \dfrac{a_{-p}}{(z - z_0)^p}$, called *principal part* (of *negative powers*, etc.).

The formulas for $a_n$ and $a_{-p}$ are similar, via the correspondence $n \leftrightsquigarrow -p$.

The integral in $a_n$ looks like the Cauchy formula for $f^{(n)}(z_0)$, but generally speaking Theorem X.1.18 cannot be applied, since $(C_R) \not\subset D$.

The coefficient $a_{-1}$ has a direct connection with the integral of $f$, namely

$$a_{-1} = \frac{1}{2\pi i}\int_{C_r} f(\zeta) d\zeta .$$

Its special utility in evaluating integrals justifies its distinguishing name:

**2.3. <u>Definition</u>.** Let us consider *D*, $z_0$ **,** $\Theta(z_0, r, R)$, and *f*, as in the above Theorem 2.1. If the hypotheses of this theorem are satisfied for arbitrarily small radiuses $r > 0$, $r < R$, then we say that the Laurent series is developed *around* $z_0$ . In this case, the coefficient $a_{-1}$ of the corresponding Laurent series is called *residue* of $f$ at $z_0$, and we note

$$a_{-1} = \mathrm{Rez}\,(f, z_0) = \mathrm{Rez}\, f\,(z_0), \text{ etc.}$$

**2.4. <u>Example</u>.** Function $f(z) = (z^2 - 3z + 2)^{-1}$ is defined on $\mathbb{C} \setminus \{1, 2\}$. To obtain its Laurent series in $\Theta(0,1,2)$, we decompose it in *simple* fractions

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1} \ .$$

Because $1 < |z| < 2$, these fractions represent sums of the geometric series

$$\frac{1}{z-2} = \frac{-1}{2}\frac{1}{1-(z/2)} = \frac{-1}{2}\left(1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \ldots\right),$$

$$\frac{1}{z-1} = \frac{1}{z}\frac{1}{1-(1/z)} = \frac{1}{z}\left(1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \ldots\right).$$

Consequently, the Laurent series of $f$ in the crown $\Theta(0,1,2)$ gives

$$f(z) = \ldots + \left(\frac{1}{z}\right)^2 + \frac{1}{z} - \frac{1}{2} - \frac{z}{4} - \frac{z^2}{8} - \ldots \ .$$

The coefficient $a_{-1} = 1$ from this Laurent series does NOT represent the residue of $f$ at $z_0 = 0$, since the development is not valid around $z_0$ . To get this residue, we write the geometric series for $|z| < 1$ (which implies $|z| < 2$),

$$\frac{1}{z-2} = \frac{-1}{2}\frac{1}{1-(z/2)} = -\frac{1}{2} - \frac{z}{4} - \frac{z^2}{8} - \ldots$$

$$\frac{1}{1-z} = 1 + z + z^2 + \ldots$$

The resulting Laurent series (around 0) has only positive powers, namely

$$f(z) = \frac{1}{2} + \frac{3}{4}z + \frac{7}{8}z^2 + \ldots$$

hence $\mathrm{Rez}\,(f, 0) = 0$.

To obtain $\mathrm{Rez}\,(f, 1)$, we consider $|z-1| < 1$ in the geometric series

$$\frac{1}{z-2} = \frac{-1}{1-(z-1)} = -1 - (z-1) - (z-1)^2 - \ldots$$

and we write the Laurent series of *f* around 1 , which is

$$f(z) = \frac{-1}{z-1} - 1 - (z-1) - (z-1)^2 - \ldots$$

Consequently, $\mathrm{Rez}\,(f, 1) = -1$.

Similarly, we find $\mathrm{Rez}\,(f, 2) = +1$.

**2.5. Remark.** We can develop a function $f: D \to \mathbb{C}$ around $z_0 \in \mathbb{C}$ in the following cases only:

1. $z_0 \in D$,
2. $z_0$ is a pole, or
3. $z_0$ is an essential singular point.

In the first case, $z_0$ is a regular point, and the Laurent series contains only positive powers; more exactly, it reduces to a Taylor series

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \ldots$$

In fact, according to the fundamental theorem X.1.11, the coefficients of the Laurent series (see Theorem 2.1) have the values

$$a_{-p} = \frac{1}{2\pi i}\int_{C_r} f(\zeta)\cdot(\zeta - z_0)^{p-1}\,d\zeta = 0\,,\ \forall p \in \mathbb{N}^*$$

and according to the Cauchy formulas X.1.18,

$$a_n = \frac{1}{2\pi i}\int_{C_R} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\,d\zeta = \frac{f^{(n)}(z_0)}{n!}\,,\ \forall n \in \mathbb{N}$$

Except Case 1, $z_0$ can be univalent isolated singular point. The principal part of the Laurent series around $z_0$ cannot vanish any more, and the single difference we can make refers to the number of terms. If the principal part of the Laurent series around $z_0$ has a finite number of terms, i.e.

$$f(z) = \frac{a_{-p}}{(z - z_0)^p} + \ldots + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \ldots + a_n(z - z_0)^n + \ldots$$

then $\lim\limits_{z \to z_0} (z - z_0)^p f(z) = a_{-p}$ is finite, hence $z_0$ is a pole (Case 2).

The remaining possibility for the principal part of the Laurent series around $z_0$ is to contain infinitely many terms. Here we recognize Case 3, when $\lim\limits_{z \to z_0} (z - z_0)^p f(z)$ does not exist, $\forall p \in \mathbb{N}^*$.

The evaluation of the residues at poles reduces to derivations:

**2.6. Proposition.** If $f: D \to \mathbb{C}$ has a pole of order $p$ at $z_0$, then

$$\text{Rez}\,(f, z_0) = \frac{1}{(p-1)!}\cdot\frac{d^{p-1}}{dz^{p-1}}\left[(z - z_0)^p \cdot f(z)\right]\Bigg|_{z \to z_0} .$$

Proof. By hypothesis, function $f$ has the development

$$f(z) = \frac{a_{-p}}{(z - z_0)^p} + \ldots + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \ldots + a_n(z - z_0)^n + \ldots$$

The resulting development of the function $\varphi(z) = (z - z_0)^p f(z)$ has Taylor coefficients, hence $a_{-1} = \left(\varphi^{(p-1)}(z_0)\right)/(p-1)!$ . ◇

**2.7. <u>Corollary</u>.** The residue of a meromorphic function, say $f = \dfrac{A}{B}$, where $A$ and $B$ are derivable on $D$, at a simple pole $z_0 \in D$, is

$$\text{Rez}\,(f, z_0) = \frac{A(z_0)}{B'(z_0)} .$$

<u>Proof</u>. It is easy to see that $f$ has a pole of order $p$ at $z_0$ iff $F = 1/f$ has a zero of the same order at this point. In our case, this means that

$$A(z_0) \neq 0,\ B(z_0) = 0, \text{ and } B'(z_0) \neq 0.$$

To complete the proof, we take $p = 1$ in the previous proposition. ◇

**2.8. <u>Examples</u>.** ($i$) Function $f(z) = \dfrac{1}{\sin z}$ has a simple pole at $z_0 = 0$ (divide power series, if not convinced). The above corollary gives Rez $(f, 0) = 1$.

($ii$) The same point $z_0 = 0$ is double pole of the function $g(z) = \dfrac{1}{\sin z^2}$. For $p = 2$, the formula from the Proposition 2.6 leads to

$$\text{Rez}\,(g, 0) = \frac{d}{dz}\left(\frac{z^2}{\sin z^2}\right)\Bigg|_{z\to 0} = \lim_{z\to 0} \frac{2z\sin z^2 - 2z^3 \cos z^2}{\sin^2 z^2} = 0.$$

In this case, the method of operating with series seems to be a simpler than deriving. In the Laurent series we may look only for $a_{-1}$ = Rez $(g, 0)$.

($iii$) To find the residue of a function at an essential singularity, we have to develop in Laurent series, and identify the coefficient $a_{-1}$ . For example,

$$h(z) = \sin\frac{1}{z} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\left(\frac{1}{z}\right)^{2k+1}$$

has an essential singularity at $z_0 = 0$, and $a_{-1}$ = Rez $(h, 0) = 1$.

**2.9. <u>Remark</u>.** As we have already mentioned in § IV.5, the classification of the singular points refers to the point at infinity too. In particular, $\infty$ can be univalent isolated singular point of a function $f: D \to \mathbb{C}$ , which means that there exists some $r > 0$ such that

$$\complement\,(\overline{C(0,r)}) \overset{not.}{=} \{z \in \mathbb{C} : |z| > r\} \subseteq D.$$

This condition shows that $\infty$ is the only singular point in a neighborhood

$$V(\infty, r) \overset{def.}{=} \complement\,(\overline{C(0,r)}) \cup \{\infty\} .$$

In the spirit of Theorem 2.1, we may interpret $\complement\,(\overline{C(0,r)})$ as a crown $\Theta(0, r, \infty)$, and the Laurent series as a development *around* $\infty$. If this series contains only negative powers of $z$, i.e. it has the form

$$f(z) = \ldots + \frac{a_{-p}}{z^p} + \ldots + \frac{a_{-1}}{z} + a_0 ,$$

then we consider that $\infty$ is a *regular* point (since $f(\infty)=a_0$ makes sense).

In the contrary case, when positive powers do exist, we consider that $\infty$ is a singular point, and we make the distinction between pole and essential singularity by the number (finite or infinite) of positive powers.

For example, in the crown $\Theta(0,2,\infty)$, the function $f(z)=(z^2-3z+2)^{-1}$ from 2.4, has the development

$$f(z)=\frac{1}{z}\frac{1}{1-(2/z)}-\frac{1}{z}\frac{1}{1-(1/z)} = \frac{1}{z^2}+\frac{3}{z^3}+\ldots$$

Consequently, $\infty$ is a regular point of $f$. On the other hand, $\infty$ is a simple pole of the function $z^2/(z-2)$, and essential singular point of $e^z$.

Recalling the final part of the Remark 2.2, we may speak of *residue at* $\infty$, which should be naturally related to the integral on $C_r$. Because the border of $\Theta(0,r,\infty)$ is $C^-(0,r)$, this "residue" should assure the relation

$$\text{Rez}\,(f,\,\infty) = \frac{1}{2\pi i}\int_{C^-(0,r)} f(\zeta)\,d\zeta\,.$$

More exactly, the expression of the Laurent coefficients in Theorem 2.1, are suggesting the following:

**2.10. <u>Definition</u>.** Let $\infty$ be univalent isolated singular point of the function $f: D\to\mathbb{C}$, and let $r > 0$ be a number for which $\Theta(0,r,\infty)\subseteq D$. If $a_{-1}$ is the coefficient of $1/z$ in the Laurent series of $f$ in $\Theta(0,r,\infty)$, then $-\,a_{-1}$ is said to be the *residue* of $f$ at $\infty$, and we note

$$\text{Rez}\,(f,\,\infty) \overset{def.}{=} -\,a_{-1}\,.$$

Alternatively, if we note $g(z) = f(1/z)$, then the change $\zeta = 1/z$ leads to

$$\int_{C^-(0,r)} f(\zeta)\,d\zeta = \int_{C(0,\frac{1}{r})} \frac{g(z)}{z^2}\,dz\,,$$

hence we may define the residue of $f$ at $\infty$ by the coefficient $b_1$ in the Laurent series of $g$ in the crown $\Theta(0,0,1/r)$.

There is no formula similar to that in Proposition 2.6 for the evaluation of the Rez $(f,\,\infty)$, so we have always to realize Laurent series around $\infty$.

**2.11. <u>Examples</u>.** (*i*) For the above function $f(z)=(z^2-3z+2)^{-1}$ (compare to 2.6 and 2.9) we have Rez $(f,\,\infty) = 0$.

(*ii*) The value Rez $(z^2/(z-2),\,\infty) = 4$ results from the development

$$z^2/(z-2) = z+2+\frac{4}{z}+\frac{8}{z^2}+\ldots$$

which holds in $\Theta(0,2,\infty)$. The variant of replacing $z\mapsto 1/z$ leads to the same result, but it is based on the development of the function $1/z(1-2z)$ in the crown $\Theta(0,0,1/2)$, namely

$$\frac{1}{z(1-2z)}=\frac{1}{z}+2+4z+8z^2+\ldots$$

(*iii*) $\infty$ is an essential singular point of $e^z$, and Rez ($e^z$, $\infty$) = 0 comes out from the very definition of the function *exp*, namely $1+\frac{z}{1!}+\frac{z^2}{2!}+\ldots$ , which makes sense in the entire $\mathbb{C}$ = $\Theta(0,0,\infty)\cup\{0\}$.

The calculus of the integrals by residues is based on the following:

**2.12. <u>Residues Theorem</u> (Cauchy).** Let $D\subseteq\mathbb{C}$ be a domain, on which the function $f : D\to\mathbb{C}$ is derivable, and let $\gamma\subset D$ be a closed (simple and piece-wise smooth) curve. If $z_1$ , $z_2$ , …, $z_n$ , are the only univalent isolated singular points from ($\gamma$), i.e. $(\gamma)\setminus\{z_1, z_2, \ldots, z_n\}\subset D$, then

$$\int_\gamma f(z)\,dz=2\pi i\sum_{k=1}^{n}\text{Rez}(f,z_k).$$

<u>Proof</u>. Let $\gamma_k = C(z_k , r_k)$, $k=\overline{1,n}$, be disjoint circles (as in Fig.X.2.2), such that $(\gamma)\setminus\left[\bigcup_{k=1}^{n}(\gamma_k)\right]\subset D$. According to Theorem X.1.16, we have

$$\int_\gamma f(z)\,dz = \sum_{k=1}^{n}\int_{\gamma_k} f(z)\,dz.$$

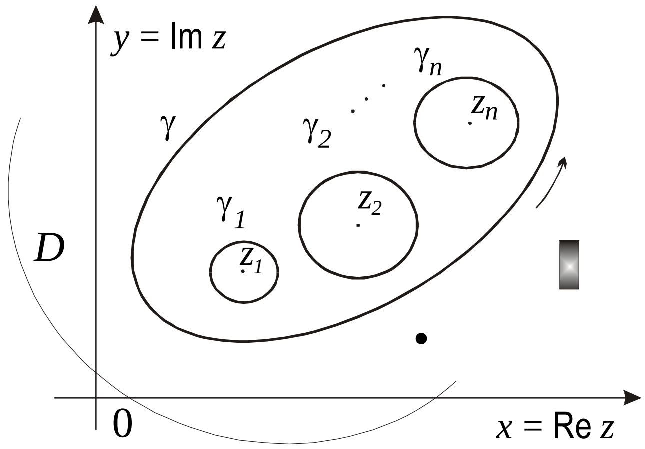



Fig. X.2.2.

By Theorem 2.1, the coefficients $a_{-1}^k$ in the Laurent developments of *f* around $z_k$, $k=\overline{1,n}$, have the values

$$a_{-1}^k=\frac{1}{2\pi i}\int_{\gamma_k} f(z)\,dz \overset{def.}{=} \text{Rez}(f, z_k).$$

It is enough to replace $\int_{\gamma_k} f(z)\,dz$ in $\int_\gamma f(z)\,dz$ . ◇

**2.13. Example.** Let us evaluate $I_n = \int_{\gamma_n} f(z)\,dz$, where $\gamma_n = C(0, n+\frac{1}{2})$, $n \in \mathbb{N}^*$, and $f(z) = \left(z^2 - 3z + 2\right)^{-1} \cdot \exp(1/z)$. The first step is to find out the singular points of $f$, and the second one is to establish what singularities are in $(\gamma_n)$; finally, we apply the Residues Theorem, and we make calculus.

Function $f$ is not definable at the points $z_0 = 0$, $z_1 = 1$, $z_2 = 2$, and $\infty$. The nature of these points is: $z_0$ is an essential singularity, $z_1$ and $z_2$ are simple poles, and $\infty$ is a regular point. Obviously, $z_0 \in (\gamma_0)$, $\{z_0, z_1\} \subset (\gamma_1)$, and $\{z_0, z_1, z_2\} \subset (\gamma_n)$ for all $n \geq 2$, since $f$ is derivable in $\complement\,(\overline{C(0,2)})$. Theorem 2.12 furnishes the following values of the integrals:

$$I_0 = 2\pi i \,\mathrm{Rez}\,(f, 0),$$
$$I_1 = 2\pi i \,[\mathrm{Rez}\,(f, 0) + \mathrm{Rez}\,(f, 1)],$$
$$I_2 = 2\pi i \,[\mathrm{Rez}\,(f, 0) + \mathrm{Rez}\,(f, 1) + \mathrm{Rez}\,(f, 2)].$$

In addition, we have $I_n = I_2$ for all $n \geq 2$, and alternatively,

$$I_2 = -2\pi i \,\mathrm{Rez}\,(f, \infty).$$

Now, we evaluate the residues. From the development in $\Theta(0,0,1)$,

$$f(z) = \left(\frac{1}{2} + \frac{3}{4}z + \frac{7}{8}z^2 + \ldots\right) \cdot \left(1 + \frac{1}{z \cdot 1!} + \frac{1}{z^2 \cdot 2!} + \ldots\right),$$

it follows that $\mathrm{Rez}\,(f, 0) = \sum_{n=1}^{\infty} \frac{2^n - 1}{2^n} \cdot \frac{1}{n!} = e - \sqrt{e}$.

Using the formulas for simple poles (see 2.6 and 2.7), we obtain

$$\mathrm{Rez}\,(f, 1) = \left.\frac{\exp(1/z)}{z-2}\right|_{z=1} = -e, \text{ and } \mathrm{Rez}\,(f, 2) = \left.\frac{\exp(1/z)}{z-1}\right|_{z=2} = \sqrt{e}.$$

Finally, because in $\Theta(0,2,\infty)$ we have

$$f(z) = \left(\frac{1}{z^2} + \frac{3}{z^3} + \ldots\right) \cdot \left(1 + \frac{1}{z \cdot 1!} + \frac{1}{z^2 \cdot 2!} + \ldots\right),$$

it follows that $\mathrm{Rez}\,(f, \infty) = 0$. To conclude, we mention the values

$$I_n = \begin{cases} 2\pi i\left(e - \sqrt{e}\right) & \text{if } n = 0 \\ -2\pi i \sqrt{e} & \text{if } n = 1 \\ 0 & \text{if } n \geq 2\,. \end{cases}$$

This example obeys the specific restriction in Theorem 2.12, which asks the singular points of $f$ be either in the interior or in the exterior part of the (closed) curve from the integral. A natural problem is to find the values of the integrals $J_n = \int_{\chi_n} f(z)\,dz$, where $\chi_n = C(0, n+1)$, or $L_n = \int_{\lambda_n} f(z)\,dz$, where $\lambda_n = \{z = x + i\,y : |x| + |y| = n+1\}$, etc. To solve such problems, we add a couple of improvements to Theorem 2.12:

**2.14. <u>Semi-Residues Theorem</u>.** Let the objects $D$, $f$, $\gamma$, and $z_1$, $z_2$, …, $z_n$ satisfy the hypotheses of Theorem 2.12, and in addition, let $z_0 \in \gamma$ be a simple pole of $f$. If $z_0$ is an *angular* point of $\gamma$, where the tangent jumps by θ radians in the positive sense of $\gamma$, then

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{n} \text{Rez}(f, z_k) + (\pi - \theta)\, i\, \text{Rez}\,(f, z_0).$$

In particular, if $z_0$ belongs to a smooth sub-arc of $\gamma$, then

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{n} \text{Rez}(f, z_k) + \pi i\, \text{Rez}\,(f, z_0)\,.$$

<u>Proof</u>. The Laurent series of $f$ around $z_0$ has the form

$$f(z) = \frac{a_{-1}}{z - z_0} + \underbrace{a_0 + a_1(z - z_0) + \ldots}_{\varphi(z)}\,,$$

where $\varphi$ is derivable in a neighborhood of $z_0$. Let $\gamma_r$ be an arc of circle in such a neighborhood, which isolates $z_0$ as in Fig.X.2.3. Obviously,

$$\Gamma_r = \gamma\big|_{AB} \cup \gamma_r$$

is a closed (simple, and piece-wise smooth) curve, on which we have

$$\int_{\Gamma_r} f(z)\,dz = 2\pi i \sum_{k=1}^{n} \text{Rez}(f, z_k)\,.$$

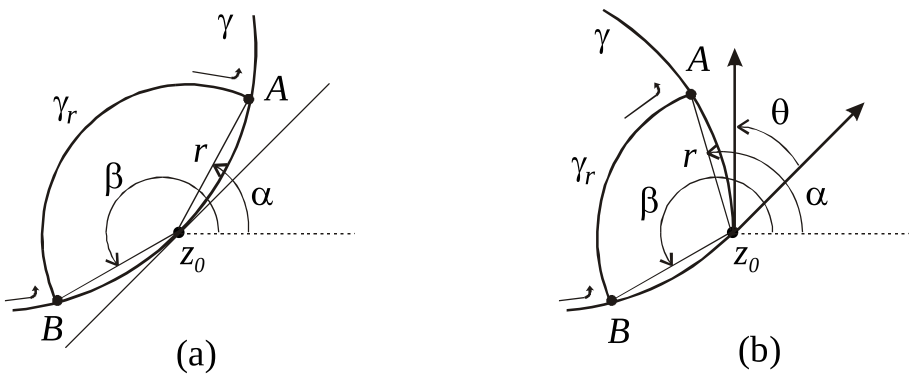



Fig.X.2.3.

On the other hand, we may decompose this integral in two parts

$$\int_{\Gamma_r} f(z)\,dz = \int_{\gamma|_{AB}} f(z)\,dz + \int_{\gamma_r} f(z)\,dz\,.$$

For the first integral we have $\lim\limits_{r\to 0}\int_{\gamma|_{AB}} f(z)\,dz = \int_\gamma f(z)\,dz$. The second one can be decomposed in two integrals according to the form of $f$, namely

$$\int_{\gamma_r} f(z)\,dz = a_{-1}\int_{\gamma_r} \frac{dz}{z - z_0} + \int_{\gamma_r} \varphi(z)\,dz\,.$$

Using the parameterization $z = z_0 + r\ e^{it}$, $t \in [\beta, \alpha]$, of $\gamma_r$, we obtain

$$\int_{\gamma_r} \frac{dz}{z - z_0} = \int_\beta^\alpha i\,dz = -(\beta - \alpha)i = -(\pi - \theta)i\,.$$

Because $\gamma_r$ is a compact set, and function $\varphi$ is continuous on $\gamma_r$, there exists $M = \sup\{|\varphi(z)| : z \in \gamma_r\} < \infty$. In addition, the length of $\gamma_r$ has the form $L = (\beta - \alpha)r$, hence the property of boundedness (X.1.7) gives

$$\left|\int_{\gamma_r} \varphi(z)\,dz\right| \le M\ (\beta - \alpha)r \underset{r\to 0}{\to} 0\ .$$

If we take $r \to 0$ in all the integrals from above, then we obtain

$$2\pi i \sum_{k=1}^{n} \text{Rez}(f, z_k) = \int_\gamma f(z)\,dz - a_{-1}(\pi - \theta)i\,,$$

where we have to replace $a_{-1} = \text{Rez}\,(f, z_0)$.

In particular, if $z_0$ belongs to a smooth sub-arc of $\gamma$, then there is no jump of the tangent, i.e. $\theta = 0$. ◇

The values of $J_n$ and $L_n$ from 2.13 are given in Problem 7 at the end.

In the final part of this section we will apply the residues theory to the real integral calculus. The main difficulty rises from the curves on which we integrate: the Residues Theorem holds on closed curves, while the real integrals are defined on parts of $\mathbb{R}$, which are non-closed curves. Therefore we are interested in constructing closed curves by adding extra curves. The problem is to control the integrals on these additional curves, which usually means that we may neglect these integrals in a limiting process:

**2.15. Jordan's Lemma #1.** Let $D \subseteq \mathbb{C}$ be a domain, on which the function $f: D \to \mathbb{C}$ is derivable, and let $z_0 \in \mathbb{C}$ be fixed. With vertex $z_0$ we consider an angle of value $\alpha$ (radians), in which $\gamma_r$ represents the arc of a circle of radius $r$, centered at $z_0$. We suppose that $\gamma_r \subset D$ holds for all $r$ in a neighborhood of $r_0$, where $r_0 \in \overline{\mathbb{R}}_+$ defines a "limit position" of $\gamma_r$, (as in Fig.X.1.4). We claim that if

$$\lim_{r\to r_0} \max_{z\in\gamma_r} |(z - z_0)\cdot f(z)| = 0$$

then we may neglect the integral on $\gamma_r$, i.e.

$$\lim_{r\to r_0} \int_{\gamma_r} f(z)\,dz = 0\ .$$

Proof. Let us remark that $M_r \overset{not.}{=} \max_{z\in\gamma_r} |(z - z_0)\cdot f(z)|$, which appears in the hypothesis, makes sense as a finite number, since $\gamma_r$ is a compact set and $f$ is a continuous function. We recall that the length of $\gamma_r$ is $L_r = \alpha\, r$. According to Theorem X.1.7, we have

$$\left|\int_{\gamma_r} f(z)\,dz\right| = \left|\int_{\gamma_r} \frac{(z-z_0)\,f(z)}{z-z_0}\,dz\right| \le \frac{M_r}{r} L_r = \alpha\, M_r \ .$$

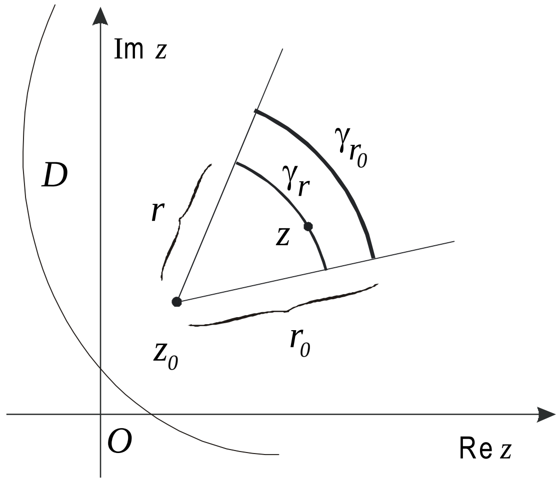



Fig.X.2.4.

Using the hypothesis that

$$\forall \varepsilon > 0 \ \exists \delta(\varepsilon) > 0 \ \textit{such that} \ \left(|r - r_0| < \delta(\varepsilon) \Rightarrow M_r < \varepsilon\right)$$

we obtain $\left|\int_{\gamma_r} f(z)\,dz\right| \le \alpha\,\varepsilon$ . ◇

**2.16. <u>Example</u>.** Jordan's Lemma #1 is useful in finding the integrals

$$I_n = \int_0^\infty \frac{dx}{1+x^n} \ , \ n \in \mathbb{N} \ , \ n \ge 2 \, .$$

As usually, the first step is to identify the singular points. In this case, function $f(z) = (1+z^n)^{-1}$ has $n$ simple poles, namely

$$z_k = \exp\left[\frac{2k+1}{n}\pi\, i\right] \ , \ k = \overline{0, n-1} \ .$$

Let $\Gamma_r = [O, A] \cup \gamma_r \cup [B, O]$ be a closed curve like in Fig.X.2.5. If we take $r > 1$ and $\alpha = 2\pi/n = 2\arg z_0$, then $z_0$ will be the only pole in $(\Gamma_r)$. Using

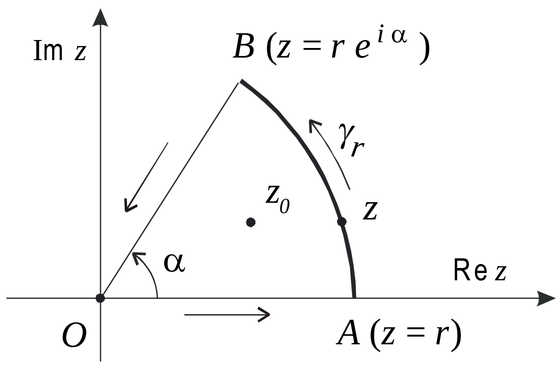



Fig.X.2.5.

the Residues Theorem, we obtain

$$\int_{\Gamma_r} f(z)\,dz = 2\pi i\,\mathrm{Rez}(f, z_0) = -\frac{2\pi i}{n} e^{\frac{\pi i}{n}}.$$

On the other hand, this integral equals a sum of integrals, namely

$$\int_{\Gamma_r} f(z)\,dz = \int_0^r \frac{dx}{1+x^n} + \int_{\gamma_r} f(z)\,dz + \int_{[B,O]} f(z)\,dz \ .$$

Following the definition of an improper integral, the first term gives

$$\lim_{r\to\infty} \int_0^r \frac{dx}{1+x^n} = I_n \ .$$

According to the above lemma, applied to $f$, at $z_0 = 0$, and $r_0 = \infty$, we may neglect the second integral, i.e. $\lim\limits_{z\to\infty} \dfrac{z}{1+z^n} = 0$ implies

$$\lim_{r\to\infty} \int_{\gamma_r} f(z)\,dz = 0.$$

Finally, using the parameterization $z = \rho e^{i\alpha}$, $\rho \in [r, 0]$, we may reduce the third integral to the first one:

$$\int_{[B,O]} f(z)\,dz = e^{i\alpha} \int_r^0 \frac{d\rho}{1+\rho^n e^{in\alpha}} = -e^{i\alpha} \int_0^r \frac{dx}{1+x^n} \ .$$

To conclude, the limit process $r \to \infty$ leads to $I_n = \pi \big/ n \sin\frac{\pi}{n}$ .

**2.17. <u>Jordan's Lemma #2</u>.** Let $D \subseteq \mathbb{C}$ be a domain, for which

$$\{z \in \mathbb{C} : \mathrm{Im} z > 0\} \subset D,$$

and let the function $f: D \to \mathbb{C}$ be derivable on $D$, except a finite number of univalent isolated singular points. If $\lim\limits_{z\to\infty} f(z) = 0$, then

$$\lim_{r\to\infty} \int_{C_r} e^{i\mu z} f(z)\,dz = 0\ ,\ \forall\, \mu > 0,$$

where $C_r$ is the upper half-circle of radius $r$, centered at 0.

<u>Proof</u>. Using the parameterization $z = re^{it}$, $t \in [0, \pi]$, we obtain

$$I_r \overset{not.}{=} \int_{C_r} e^{i\mu z} f(z)\,dz = \int_0^\pi e^{i\mu r(\cos t + i\sin t)} f(r\,e^{it})\, r i e^{it}\,dt.$$

If $r$ is great enough to include all singularities under $C_r$, then we may note $M_r \overset{not.}{=} \max\{|f(z)| : z \in C_r\}$, since $f$ is continuous on the compact set $C_r$. The property of boundedness for integrals on intervals of $\mathbb{R}$, gives

$$|I_r| \le r \int_0^\pi e^{-\mu r \sin t} M_r\,dt \ .$$

Because function *sin* is symmetric relative to $\frac{\pi}{2}$, this inequality becomes

$$|I_r| \le 2rM_r \int_0^{\frac{\pi}{2}} e^{-\mu r \sin t}\,dt.$$

Taking into account that $\sin t \geq \frac{2}{\pi} t$ at each $t \in [0, \frac{\pi}{2}]$, we obtain

$$\left|I_r\right| \leq 2rM_r \int_0^{\frac{\pi}{2}} e^{-\mu r \frac{2}{\pi} t}\, dt = \frac{\pi M_r}{\mu} \cdot \left[1 - e^{-\mu r}\right] < \frac{\pi M_r}{\mu} .$$

By hypothesis, $r \to \infty$ implies $M_r \to 0$. ◇

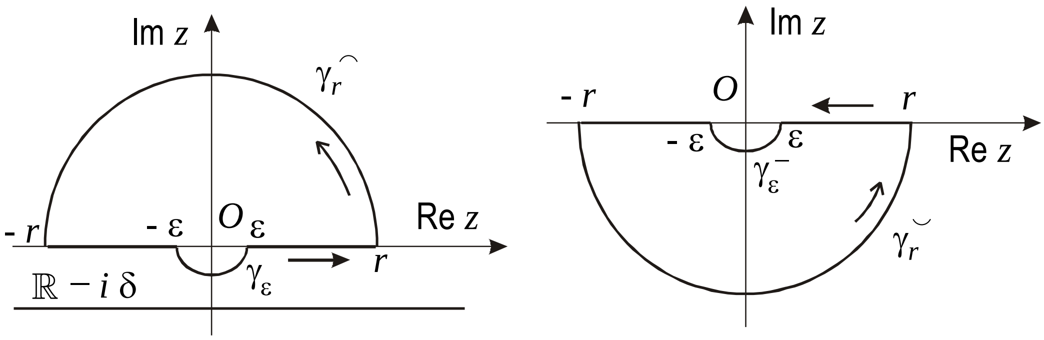
**2.18. <u>Example</u>.** Let us show that the Heaviside's function

$$\eta(t) = \begin{cases} 0 & if \;\; t < 0 \\ 1 & if \;\; t > 0 \end{cases}$$

allows an *integral representation*, expressed by the following formula

$$\eta(t) = \frac{1}{2\pi i} \int_{\mathbb{R} - i\delta} \frac{e^{itz}}{z} dz \;, \; \forall \delta > 0 \;.$$

To prove this equality, we first remark that the integral preserves its value if we replace $\mathbb{R} - i\,\delta$ by $\Lambda = (-\infty, -\varepsilon] \cup \gamma_\varepsilon \cup [\varepsilon, +\infty)$. The definition of this




*a*) *b*)

Fig.X.2.6.

improper integral involves the curve $\Lambda_r = (-r, -\varepsilon] \cup \gamma_\varepsilon \cup [\varepsilon, +r)$, since

$$I \overset{not.}{=} \int_{\mathbb{R} - i\delta} \frac{e^{itz}}{z} dz = \int_\Lambda \frac{e^{itz}}{z} dz = \lim_{r \to \infty} \int_{\Lambda_r} \frac{e^{itz}}{z} dz \;.$$

We construct the closed curve, which we need in the residues theorem, in different ways, depending on sign $t$ (compare Fig.X.2.6 a) and b)).

<u>Case</u> a) $t > 0$. If we note $\Gamma_r^{\frown} = \Lambda_r \cup \gamma_r^{\frown}$, then we obtain

$$\int_{\Gamma_r^{\frown}} \frac{e^{itz}}{z} dz = 2\pi i \operatorname{Rez}\left(\frac{e^{itz}}{z}, 0\right) = 2\pi i \;.$$

<u>Case</u> b) $t < 0$. The interior of $\Gamma_r^{\smile} = \Lambda_r^- \cup \gamma_r^{\smile}$ contains no singularity, hence

$$\int_{\Gamma_r^{\smile}} \frac{e^{itz}}{z} dz = 0.$$

Using the Jordan's Lemma #2, we neglect the integrals on $\gamma_r^{\smile}$ and $\gamma_r^{\frown}$.

**2.19. <u>Classes of real integrals</u>.** To be calculable by residues, the real integrals shall have certain particular forms. In practice we have to identify the *class* the given integral belongs to, and to apply the specific technique.
**A. Improper integrals of rational functions.** Let us analyze the integral

$$I = \int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)} dx \ ,$$

where $P$ and $Q$ are polynomials with real coefficients. The existence of this integral (see § V.2) is assured if

- $Q$ has no real roots, and
- grad $Q \geq$ grad $P + 2$ .

We suppose that these conditions are satisfied, and we pursue the technique of calculating its value.

The starting point is the very definition of an improper integral, which (in this case) allows the form $I = \lim_{r \to \infty} I_r$, where

$$I_r = \int_{-r}^{+r} \frac{P(x)}{Q(x)} dx \ .$$

To apply the Residues Theorem, we construct the closed curve

$$\Gamma_r = [-r,+r] \cup \gamma_r \ ,$$

where $\gamma_r$ is a half-circle of radius $r$, centered at 0 (as in Fig.X.2.7).

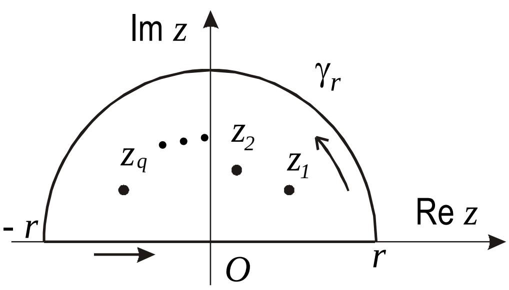



Fig.X.2.7.

Let $z_1, z_2, \dots , z_q$ be the roots of $Q$ , where grad $Q = 2q$, in the upper half-plane $\{z \in \mathbb{C} : \text{Im}z > 0\}$. These points are poles of the complex function $f$ , of values $f(z) = \dfrac{P(z)}{Q(z)}$ . If $r$ is great enough, all these poles are in $(\Gamma_r)$, hence

$$\int_{\Gamma_r} f(z)\, dz = 2\pi i \sum_{k=1}^{q} \text{Rez}\,(f, z_k)\,,$$

where the right-hand member does not depend on $r$. What remains is to decompose this integral on the sub-arcs $[-r,+r]$ and $\gamma_r$, and to take $r \to \infty$, since the Jordan's Lemma #1 operates on $\gamma_r$ .

**B. Integrals of rational functions in *sin* and *cos*.** Let us evaluate

$$J = \int_0^{2\pi} \mathscr{R}(\sin t, \cos t)\, dt\,,$$

where $\mathscr{R}$ is a (real) rational function. Based on the Euler's relations

$$\sin t = \frac{e^{it} - e^{-it}}{2i}\ ,\ \cos t = \frac{e^{it} + e^{-it}}{2}\ ,$$

we may change the variable $t \mapsto z = e^{it}$, $t \in [0, 2\pi]$. Because this change of variables represents a parameterization of the unit circle, we obtain

$$J = \int_{C(0,1)} \frac{P(z)}{Q(z)}\, dz\,,$$

where $P$ and $Q$ are polynomials with real coefficients. The value of $J$ comes out by the Residues Theorem for the poles in the interior part of $C\,(0,\,1)$.

**C. Improper Integrals of a rational function times *cos* (or times *sin*).** Let us consider integrals of the form

$$K_c = \int_{-\infty}^{+\infty} R(x) \cdot \cos \mu x\, dx\ ,\ K_s = \int_{-\infty}^{+\infty} R(x) \cdot \sin \mu x\, dx\ ,$$

where $R = P/Q$ is a rational function for which $\operatorname{grad} P < \operatorname{grad} Q$, and $\mu > 0$. In addition we suppose that $Q$ has no roots in $\mathbb{R}$, and $R$ is an even function in $K_c$, respectively odd in $K_s$. In practice, if $K_c$ is given, possibly on $[0, \infty)$, then $K_s = 0$ because of parity, and vice versa. Therefore we may combine $K_c$ and $K_s$ in a complex integral of the form

$$K = \int_{\mathbb{R}} e^{i\mu x} R(x)\, dx \overset{def.}{=} \lim_{r \to \infty} \int_{-r}^{r} e^{i\mu x} R(x)\, dx\ .$$

To obtain a closed curve, we construct $\Gamma_r = [-r, r] \cup C_r$, where $C_r$ is the upper half-circle of radius $r$, centered at 0 (as in Lemma 2.17). Let us note by $z_1, z_2, \ldots, z_q$ the roots of $Q$ in the upper half-plane (compare to class A). These points are poles of the integrated function, which are contained in the interior of $\Gamma_r$ whenever $r > \max\{|z_k| : k = \overline{1,q}\}$. Theorem 2.12 gives

$$\int_{\Gamma_r} e^{i\mu z} R(z)\, dz = 2\pi i \sum_{k=1}^{q} \operatorname{Rez}(e^{i\mu z} R(z), z_k)\,,$$

where the right-hand term is independent of $r$. If we decompose

$$\int_{\Gamma_r} e^{i\mu z} R(z)\, dz = \int_{-r}^{r} e^{i\mu x} R(x)\, dx + \int_{C_r} e^{i\mu z} R(z)\, dz\,,$$

then the limit $r \to \infty$ avoids the integral (see Lemma #2), and we obtain

$$K = 2\pi i \sum_{k=1}^{q} \operatorname{Rez}(e^{i\mu z} R(z), z_k)\,.$$

The reader may find examples of such integrals at the end of the section. There are many other types of integrals (e.g. with multi-valued functions), for which we recommend a larger bibliography. To be more convincing about the advantages of using the Cauchy theory, we conclude by several *remarkable* real integrals, which are easily obtained by residues techniques.

**2.20. Remarkable integrals.** (a) The **Poisson**'s integral is defined by

$$P = \int_0^\infty \frac{\sin t}{t} dt \ .$$

Using the parity of ($\sin t$) / $t$, and changing the variable, we obtain

$$P = P(\mu) = \int_0^\infty \frac{\sin \mu t}{t} dt \ = \frac{1}{2}\int_{\mathbb{R}} \frac{\sin \mu t}{t} dt .$$

According to the Euler's formula for *sin*, the problem reduces to the Heaviside's integral, and the result is $P = \pi / 2$ .

(b) There is a remarkable integral of class C in 2.19, namely

$$L = \int_0^\infty \frac{\cos t}{t^2 + \alpha^2} dt \ , \alpha > 0,$$

which is known as **Laplace**'s integral. Combining with the corresponding integral of *sin*, we obtain

$$L = \frac{1}{2}\int_{\mathbb{R}} \frac{e^{it}}{t^2 + \alpha^2} dt \ .$$

Because $i\alpha$ is the only (simple) pole in the upper half-plane, the study of the above mentioned class C leads to the value

$$L = \pi i \,\mathrm{Rez}\left(\frac{e^{iz}}{z^2 + \alpha^2}, i\alpha\right) \ = \frac{\pi}{2\alpha} e^{-\alpha} .$$

(c) Knowing the Gauss' integral $G = \int_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$ , we can evaluate

$$F = \int_0^\infty \cos^2 x \, dx \ = \int_0^\infty \sin^2 x \, dx \ = \frac{1}{2}\sqrt{\frac{\pi}{2}} \ ,$$

which are called **Fresnel**'s integrals. In fact, if $\Gamma_r$ is the contour from 2.16, for $\alpha = \pi / 4$, then $\int_{\Gamma_r} e^{iz^2} dz \ = 0$. Integrating by parts in the corresponding real integral, it follows (without Lemmas #1 or #2) that $\int_{\gamma_r} e^{iz^2} dz \underset{r\to\infty}{\longrightarrow} 0$.

The integral on [$O$, $A$] tends to $G$, and a change of variables in the integral on [$B$, $O$] leads to $G$ too.

**PROBLEMS §X.2.**

**1.** Write Laurent series for the following functions in the specified crowns:
(1) $\left(z^3-3z^2+2z\right)^{-1}$, $\Theta(0,1,2)$; (2) $\dfrac{e^{\frac{1}{z}}}{z^2+1}$, $\Theta(0,0,1)$; (3) $\sin\dfrac{z+1}{z-1}$, $\Theta(1,0,\infty)$.
Hint. (1) Compare to 2.4. (2) Multiply the series of *exp* by a geometric one. (3) Decompose $\sin\left(1+\dfrac{2}{z-1}\right)$, then write the series of *sin* and *cos*.

**2.** Develop the following functions in Laurent series around $\infty$:

$$(i)\ \frac{z^2-1}{z^2+1};\quad (ii)\ \frac{z^{99}}{z^{100}-1};\quad (iii)\ \sin^{-1}(z)^{-1}.$$

Hint. $(i)$ $1-\dfrac{2}{z^2}\cdot\dfrac{1}{1+(1/z^2)}$ ; $(ii)$ $\dfrac{1}{z}\cdot\dfrac{1}{1-(1/z^{100})}$; $(iii)$ Transform the series of $\sin^{-1}z$ around 0.

**3.** Find the residues of the following functions at the specified points:
(a) Rez $(\dfrac{z-1}{z^3-2z^2+z-2}, i)$; (b) Rez $(\sin^{-2}z, 0)$; (c) Rez $(\dfrac{\sin z}{z^4}, 0)$;
(d) Rez $(\sin\dfrac{1}{1-z}, 1)$; (e) Rez $(\sin\dfrac{z}{1-z}, 0)$; (f) Rez $(\exp(1/(z-1)), \infty)$;
(g) Rez $(\sin\dfrac{1}{z-1}, \infty)$; (h) Rez $(\dfrac{\sin^2 z}{z^5}, \infty)$; (i) Rez $(\dfrac{\cos 2\sqrt{z}}{1-\cosh\sqrt{z}}, 0)$.
Hint. Directly use 2.6, 2.7 in (a) and (b). (c) Prolong $\dfrac{\sin z}{z}$ to a derivable function. For (d) and (e), use the definition of *sin*. (f) In the series of *exp*, each term is the sum of a series; identify the coefficient of $1/z$. (g) Similarly to (f), use the series of *sin* and identify the coefficient of $1/z$ in

$$\sin\frac{\frac{1}{z}}{1-\frac{1}{z}}=\sin\left(\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\dots\right).$$

(h) Use the development $\dfrac{1-\cos 2z}{z^5}=\dfrac{2}{z^3}-\dfrac{2}{3}\cdot\dfrac{1}{z}+\dfrac{4}{45}z-\dots$ . (i) Divide the series of $\cos 2\sqrt{z}$ by that of $1-\cosh\sqrt{z}$ .

**4.** Study whether the residues always vanish at regular points. Establish the nature of $\infty$, and (if any) find the corresponding residue of the functions:

$$(az^2+bz+c)^{-1};\ 1/z;\ e^z;\ \sin^{-1}z;\ e^{\frac{1}{z}};\ \sin(1/z);\ \sqrt{z}.$$

Hint. According to Remark 2.5, Rez $(f, z_0) = 0$ whenever $z_0$, is a regular point at finite distance. Otherwise, Rez $(f, \infty)$ is not necessarily null, e.g. Rez $(1/z, \infty) = -1$. Rez $(\sin^{-1} z, \infty)$ and Rez $(\sqrt{z}, \infty)$ make no sense.

**5.** Use the Residues Theorem to evaluate the complex integrals:

$$\int_{C(0,4)} \frac{dz}{z^2 \sin z}; \int_{C(0,\frac{1}{2})} \frac{e^{\frac{1}{z}}}{(1-z)^2} dz; \int_{C(0,1)} z^2 e^{\frac{1}{1-2z}} dz; \int_{C(0,1)} z^n \cosh\tfrac{1}{z} dz.$$

Hint. Find the poles and the essential singularities in the interior part of the mentioned circles, and evaluate the corresponding residues.

**6.** Let $f: D \to \mathbb{C}$ be a derivable function, which has only a finite number of univalent isolated singular points, say $z_1, z_2, \ldots, z_n$. Show that

$$\text{Rez}(f, \infty) + \sum_{k=1}^{n} \text{Rez}(f, z_k) = 0.$$

Use this fact to evaluate $I = \int_{C(z_0,r)} \frac{dz}{z^{10}-1}$, where $z_0 = -0.001$, and $r = 1$.

Hint. If $R > \max\{|z_k| : k = \overline{1,n}\}$, then $C(0, R) \subset D = \mathbb{C} \setminus \{z_1, z_2, \ldots, z_n\}$, so the Residues Theorem gives

$$\int_{C(0,R)} f(z)\, dz = 2\pi i \sum_{k=1}^{n} \text{Rez}(f, z_k).$$

Compare to the definition of the residue at $\infty$, which shows that

$$2\pi i \,\text{Rez}(f, \infty) = \int_{C^-(0,R)} f(z)\, dz .$$

Function $f(z) = \left(z^{10}-1\right)^{-1}$, from $I$, has ten simple poles, namely the roots of order 10 of 1, $z_k = \cos\frac{2k\pi}{10} + i \sin\frac{2k\pi}{10}$, $k = \overline{0,9}$. Except $z_0 = 1$, the other nine poles lie in the interior of $C(-0.001, 1)$. Instead of evaluating the nine residues at these poles, it is easier to write $I$ = Rez $(f, \infty)$ + Rez $(f, 1)$.

**7.** Evaluate the integrals $J_n$ and $L_n$ from Example 2.13, $\forall n \in \mathbb{N}$.

Hint. The circles $\chi_n = C(0, n+1)$ from $J_n$ are smooth curves, hence we use 2.14 with $\theta = 0$. The curves $\lambda_n$ in $L_n$ are squares, hence we take $\theta = \pi/2$. The calculus indicated by Theorems 2.12 and 2.14 leads to the values

$$J_n = \begin{cases} \pi i\left(e - 2\sqrt{e}\right) & if \ n = 0 \\ -\pi i \sqrt{e} & if \ n = 1 \\ 0 & if \ n \geq 2 \end{cases} \quad \text{and} \quad L_n = \begin{cases} \pi i\left(\frac{3}{2}e - 2\sqrt{e}\right) & if \ n = 0 \\ -\frac{3}{2}\pi i \sqrt{e} & if \ n = 1 \\ 0 & if \ n \geq 2 . \end{cases}$$

**8.** Evaluate the improper integrals

$$I = \int_0^\infty \frac{x^2+1}{x^4+1}dx; \quad J = \int_{-\infty}^{+\infty} \frac{dx}{(x^2+a^2)^2}, \ a > 0.$$

Hint. Use the method described in 2.19 A. To find $I = \pi/\sqrt{2}$, evaluate the residues at the simple poles $\frac{1\pm i}{\sqrt{2}}$ . In the upper half-plane, the function from $J$ has a double pole at $ai$.

**9.** Find the values of the following real integrals by means of residues:

$$I = \int_0^\pi \frac{dt}{\sqrt{2}+\cos 2t}; \quad J = \int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta+a^2}, \ a \in (0,1).$$

Hint. Apply the scheme from 2.19 B. For $I$, the only (simple) pole in the unit circle is $1 - \sqrt{2}$ . For $J$, the corresponding pole is $a$ .

**10.** Evaluate the integrals

$$I = \int_0^\infty \frac{\cos x}{(x^2+1)^2}dx; \quad J = \int_0^\infty \frac{x\sin ax}{x^2+b^2}dx, \ a,b \in \mathbb{R}.$$

Hint. The integrals belong to class 2.19 C. In the case of $I$ there is a double pole at $i$. $J$ vanishes for $a$ = 0, and it $J$ reduces to the Poisson's integral for $b$ = 0, hence we may restrict the problem to $a$, $b$ > 0.

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