R E S P O N S E T O L A U D A T I O

I would like to begin by thanking you for the great honor. Especially since your department has as a faculty member one of the most prominent mathematicianS in my area, Professor Vicentiu Radulescu. Many people around the world, including myself, are deeply indebted to him.

I will use this opportunity to share with you some thoughts about the nature and goals of the science of mathematics. You see, the older we get, the more philosophical we become. Maybe because we want to justify what we have been doing in all our life.

If we are asked to define what is mathematics, a simple and rather naive definition, would be to say that "Mathematics is the science of quantity and space". The science of quantity in its primitive form is known as "arithmetic", while the science of space is known as "geometry".

Geometry first appeared in ancient Mesopotamia and ancient Egypt, since the people there needed to redistribute the land when the waters of the flooded rivers receded.

However, mathematics is much more than this. It is primarily a "deductive (productive) science". This was established very convincingly in 300 BC by Euclid with his famous book "Elements". In mathematics in general, starting from a set of basic ideas and notions which are considered to be self-evident and using certain rules and logical arguments, we build a whole construction of conclusions whose complexity increases continuously.

For example, in geometry it is not only the spatial point of view of the subject that matters, but also the methodology with which starting from a hypothesis we reach a conclusion. This is known in mathematics as "proof". In fact, mathematics is the one and only science characterized by the notion of rigorous proof. Proofs are the essence of mathematics. Historically the first proof was given by "Θαλής ὁ Μιλήσιος" (around 600 BC), who proved that the diameter divides the circle into two equal parts. In its beginning with Θαλής, geometry was based on the intuitive notions of "point" and "line". Three hundred years later Euclid, produced an axiomatic basis for the subject.

It is worth drawing a parallel with set theory which is in the heart of modern mathematical analysis. The corresponding development for set theory did not take 300 years but only 39 years. The " Θαλής " of set theory is Cantor, with his revolutionary ideas about infinity (starting in 1879). The "Euclid" of set theory is Ernest Zermelo, who in 1908 introduced the axioms of set theory.

Euclid introduced certain properties of points and lines and required that every theorem must be derived from them and not using arbitrary intuitive arguments. Similarly, in axiomatic set theory, set is a primitive concept which satisfies a given list of axioms. Of course, these axioms are not chosen arbitrarily, but they are an outgrowth of our intuition. However, at this point the role of intuition is terminated. Only conclusions based on these axioms are acceptable. Whether the objects described by these axioms do exist in the real world, is irrelevant for the process of formal production.

In the history of geometry, an axiom played a special role. This is the "axiom of parallel lines". A common way to formulate this axiom is the following:

> "Given a line and a point not belonging to this line, there is one and only one line passing from the point and parallel to the given one."

The problem with this axiom is that it is not self-evident. All the lines we draw on a piece of paper or on the blackboard have finite length and there is no way to know how the two lines behave when drawn indefinitely. So, by its nature, this axiom cannot be verified by observation, using only our senses. However, as we all know it is crucial in the development of the Euclidian Geometry.

In set theory we encounter a similar problem. There is an axiom in axiomatic set theory, which some mathematicians have difficulty accepting. This is the so-called "axiom of choice". This axiom can be formulated as follows:

> "The Cartesian product of a nonempty family of nonempty sets is a nonempty set" (that is, if $\{A_i\}_{i\in I}$ is a family of sets such that $I \neq \varnothing$ and $A_i \neq \varnothing$ for each $i\in I$, then the family admits a choice function, meaning that there is a function $f : I \to \cup_{i\in I} A_i$ such that $f(i)\in A_i$ for each $i\in I$)."

Most mathematicians find the axiom of choice and of course the axiom of parallel lines, intuitively very plausible. If we accept these axioms, then this is simply an act of faith. However, both axioms have profound consequences in the corresponding theories. For example, the seemingly innocent axiom of choice leads to the transfinite induction.

Several mathematicians did not feel comfortable with the axiom of choice and tried to avoid it. For this reason, it was introduced the "Limited Set Theory" (LST for short), which is the Zermelo axiomatic set theory minus the axiom of choice.

The Austrian logician Kurt Gödel (1938) proved that if the LST is consistent, then the Zermelo axiomatic set theory is. That is, Gödel showed that the axiom of choice is not more dangerous than the other axioms of set theory.

Similarly for the "Continuum Hypothesis" (CH for short), which tormented Cantor all his life. One way to state this is the following:

"Every infinite subset of $\mathbb{R}$ is either countable or has cardinal number equal to that of $\mathbb{R}$."

Gödel proved that if then CH⊕LST lead to a contradiction, then there is a contradiction hidden in the Zermelo axiomatic set theory. This means that the CH cannot be refuted. But the question remains. Can we prove the CH? Cantor all of his life tried to do this but failed. Finally, Paul Cohen (1963) established that the CH cannot be proved.

In geometry, the removal of the axiom of parallel lines leads to two new and perfectly consistent noneuclidean geometries:

- the hyperbolic geometry of Lobachevsky (1828) and Bolyai (1830);
- the elliptic geometry of Riemann (1854).

These geometries are consistent if the Euclidean geometry is. The question of consistency of the Euclidean geometry was proved at the beginning of the 20th century by David Hilbert, through the arithmetization of the geometric notions. In this way he showed that the axioms of the Euclidean geometry lead to a contradiction if and only if the rules of real numbers (that is, the rules of arithmetic) lead to a contradiction.

Of course, nobody seriously doubted the validity of the Euclidean geometry. In contrast many prominent mathematicians like Brouwer, Weyl, Poincaré to mention a few, expressed strong reservations about the axiom of choice.

In general, on the foundations of mathematics, we can detect three main currents:

- Platonism
- Formalism
- Intuitionism (or constructivism)

According to Platonism, mathematical objects are real and exist in the world of "Ideas". Their existence is a real fact, completely independent of our knowledge about them. So, according to Platonism, mathematicians discover things. Gödel and Cantor were platonists. In fact, Cantor's theory of infinity was influenced by the scholastic philosophy of the medieval theologian St. Thomas Aquinas (1225-1274), who tried to base the catholic dogma on the Aristotelian logic.

Formalism does not claim that mathematical objects are real, simply mathematics is a collection of axioms, definitions and theorems (that is, a set of forms), and there are rules which allow us to pass from one form to the other. There are no real objects. Only a language and forms. So, mathematicians invent.

Platonists and Formalists may be in opposite sides on the issue of existence, but they agree on the rules and logical principles which are allowed in the daily practice of mathematics.

In contrast, intuitionists consider real mathematics only those that can be generated by a finite structure. The two prominent advocates of this current are L. Kronecker (1823-1891) and L.E.J. Brouwer (1881-1966). Kronecker was the nemesis of Cantor. To stress his approach to mathematics, Krnecker said

"God created the integers. The rest are man made."

Brouwer formulated intuitionism as a complete current on the foundation of mathematics. For Brouwer, neither language nor logic is a presupposition for mathematics. The source of mathematics is intuition (hence the name “intuitionism”), Brouwer refused to accept the basic principle of Aristotelian logic, that of “trichotomy”, or of “middle third”.

According to this principle, for any sentence A, either A or -A (the negation of A) is true. For Brouwer and the intuitionists, this is not true. There is a “third” possibility, a “middle”, which cannot be excluded. So, for the intuitionists, proof by contradiction is not valid. Needless to say, for them, Cantor’s theory of the infinity is totally unacceptable. Such an approach to mathematics lead to a dry science without any stimulating juices. Hilbert reacted to this suffocating approach in mathematics by saying that the rejection by the intuitionists of some of the basic tools of mathematical thinking, is like prohibiting a boxer to user his/her fists.

Hilbert was very upset with these ideas of Brouwer, and essentially declared war to him. Brouwer called Hilbert “my enemy”.

There are two particular instances where the clash reached unprecedented levels.

The first occasion was on the issue of participation in the Bologna International

Congress of Mathematicians (1928) This was the first time after World War I where Germany was officially invited to participate in an international meeting. Some prominent and ultranationalist German mathematicians like Bieberbach, Kneser, Schmidt, were against accepting the invitation. Another person who felt strongly that the invitation should be rejected was Brouwer.

Although Dutch, Brouwer was strongly pro-German nationalist, so, he campaigned strongly against participating in the congress. Hilbert (although with serious health problems) tried very intensely in favor of participation and finally led a delegation of 67 German mathematicians to the meeting. He was received with a standing ovation.

The second occasion was the attempt of Hilbert to remove Brouwer from the editorial board of the prestigious journal “Mathematische Annalen”. This bitter fight lasted two years (1929-1930) and involved many prominent figures like Carathéodory and Einstein. Finally, Hilbert succeeded in removing Brouwer from the journal.

Hilbert felt very strongly about Cantor's theory of infinity. He has been quoted as saying: "Nobody will remove us from the paradise that Cantor created for us"

Hilbert viewed Cantor's theory as the finest product of mathematical genius.

Today, the overwhelming majority of mathematicians are either Platonists or formalists. In the weekdays when we work on our problems, we are confident that we are dealing with real things. In the weekend, when we have time to reflect on what we are doing, then sometimes, we reach the conclusion that our work is in fact a very elegant game of forms.

What about Applied Mathematics? Today, a lot of emphasis is put on applications because of funding reasons. Everyone tries to show that what he/she is doing has applications. In fact, very often applications are the litmus test to judge a mathematical work.

However, in spite of the importance of applications, these must never he made the measure of value of a mathematical work.

Once the famous French mathematician, J. Fourier (1768-1830), said that the purpose of mathematics lies in the explanation of physical phenomena. To this, the equally famous German mathematician, C. Jacobi (1804-1851) reacted by saying: "A philosopher like Fourier should know that the glory of the human spirit is the sole aim of all science".

Mathematics is a dynamic science changing continuously and has a promising future, provided, it resists the dictatorship of one-sided obsessions such as the current one about applications.

Paraphrasing Plato, we can say:

"Mathematics aims at the knowledge of the eternal."

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